# Short-Period Communication and the Role of Zero-Order Holding in Networked Control Systems

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***Abstract*—We discuss controller design for a networked control system (NCS) in which a stochastic linear time invariant (LTI) plant communicates with a controller over a shared medium. The medium supports a limited number of simultaneous connections between the controller and the plant's sensors and actuators, possibly subject to transmission delays. We restrict communication to periodic medium access sequences which preserve the structural properties of the plant, thus decoupling the selection of the communication from that of the controller. Using the plant's controllability/observability indices as a guide for allocating access, we show that the period of the sequences in question can be shorter than previously established. In addition, we explore the use of sequences designed for a simple NCS model, in which sensors and actuators are "ignored" by the controller when they are not actively communicating, in a more complex, but practical, setting that includes zero-order holding. We include a numerical experiment that illustrates our results in the context of LQG control.**

***Index Terms*—Networked control systems, LQG, zero-order hold (ZOH), communication sequence, controllability indices.**

## I. Introduction

This note is concerned with Networked Control Systems (NCSs) in which a linear system and its controller communicate over a shared medium which cannot accommodate all of the plant's sensors and actuators simultaneously. This situation, referred to in the literature as a *medium access constraint*, arises "naturally" in simple laboratory-scale networks [1], [2] and more sophisticated Fieldbus and CAN-based networks [3], among others. In that context, it is only meaningful to specify a controller in conjunction with a communication policy [4], [5] which prescribes the times at which the plant's sensors and actuators are to be granted medium access. The choice of communication influences the performance of the controller, and vise versa, leading to high complexity if one insists on seeking jointly optimal solutions [6]–[8]. One way to manage that complexity is to make strong assumptions regarding the underlying plant, such as block-diagonal dynamics [9], or "one-sided" access constraints [10], [11]. This paper, an extended version of [12] with detailed proofs and additional numerical results, is a continuation of a complementary approach to NCS controller design [13]–[16], that relaxes the requirement for optimal communication for the sake of being able to compose straightforward solutions.

Our approach to controller design for linear NCS with medium access constraints and delays is based on "decoupling" the control and communication subproblems. This is accomplished by: 1) restricting communication to periodic sequences, and 2) designing a controller for the resulting periodic NCS using existing techniques. The last step requires that the communication policy be such that it preserves the structural properties (e.g., controllability and observability) of the underlying plant in the presence of communication constraints. We explore the selection of such policies under an NCS model that forgoes the use of ZOH, so that the controller ignores sensors and the plant turns off actuators that are not actively communicating. Doing so makes the design of effective communication especially straightforward; here, policies are constructed based on the plant's controllability (observability) indices, in a way that allows us to establish a least conservative upper bound on their period. That bound, to be compared with recently reported results [16] and [17], is the dimension of the state vector. Furthermore, under some conditions on the plant's parameters, sequences designed for the non-ZOH model are also effective in NCSs which include ZOH elements. Being able to use the same communication pattern in both settings also allows for direct comparisons between the two, showing that the inclusion of ZOH does not always lead to better LQG performance.

For the NCSs we have in mind here, the basic constraint is lack of simultaneous access to all sensors and actuators. This "bottleneck" is handled via time-multiplexing. In particular, time on the shared medium is divided into "slots," and a so-called *communication sequence* [4] specifies which sensors/actuators are to be granted medium access during each slot. This time-division viewpoint is at the center of Fieldbus and CAN-based networks [3]. For example, the fly-by-wire system of the JAS 39 Gripen fighter plane contains a time-division-based network with 30 sensors and 11 actuator modules, all vying for approximately 1200 time slots per second on the communication bus [18]. Related work in the same setting includes [5], [9] on stabilization, [7], [8] on optimal control, and [19]. Other works with complementary approaches on NCS stabilization, estimation, and effects of data losses include [20]–[23], and [24], [25] respectively. See [26] for a fuller review. The problem of LQG design for NCS with medium access constraints is discussed in [14], [27]. NCSs with delays are explored in some detail in [28] and later extensions [29].

In the next section, we review an NCS model—without the ZOH elements typically attached to the plant's inputs and outputs—on which we later impose periodic communication. Section III shows how to choose communication sequences that preserve the structural properties of the NCS, and discusses their period and their effectiveness when a ZOH *is* included. Section IV discusses a numerical example involving LQG control of an NCS with access constraints and known delays.

## II. NCS Model and Problem Formulation

Our NCS model, illustrated in Fig. 1, follows closely that of [16]. We will take $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, and $y \in \mathbb{R}^p$ to be the plant's state, input, and output vectors, respectively. The communication medium imposes an upper bound, $w_\sigma < p$, on the number of sensors, and $w_\rho < m$ on the actuators which may communicate simultaneously with the controller; as a result, medium access must be time-multiplexed. Controller-plant communication may also be subject to transmission delays which we will take to be known. We will not consider quantization or bit-rate constraints.

The linear time invariant (LTI) plant may evolve either in discrete or in continuous time, in which case it is sampled periodically. We have then

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) + v(k) \\ y(k) &= Cx(k) + w(k) \\ k &= 0, 1, 2, \ldots \end{aligned} \tag{1}$$

where $v(\cdot)$, $w(\cdot)$, are both Gaussian, independent and identically distributed (i.i.d.), with $v(\cdot) \sim N(0, G)$, $G = G^T > 0$, and $w(\cdot) \sim N(0, I)$, and with $x(0) \sim N(x_0, \Sigma_0)$, $\Sigma = \Sigma^T > 0$. We want to find a medium access policy for the plant's sensors and actuators, and a control policy which together achieve some objective (e.g., minimize an LQG-type cost), subject to the access constraints $(w_\rho, w_\sigma)$ and delays $(\tau_{pc}, \tau_{cp})$. We begin by discussing NCSs with discrete-time, deterministic, delay-free plants.

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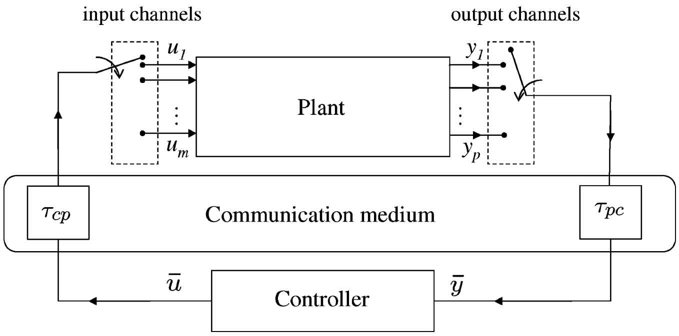



Fig. 1. A basic NCS model. The communication medium cannot facilitate simultaneous communication between all sensors/actuators and the controller; it may also impose transmission delays between plant and controller. The "open" or "closed" status of the switches indicates the medium access status of the corresponding sensors or actuators.

We will use the notion of a *communication sequence* [4], [5], [15] to describe the medium access status of the plant's inputs and outputs in (1).

*Definition 1:* Let $M, N \in \mathbb{N}$ with $N \leq M$. An $M$-to-$N$ *communication sequence* is a map, $\sigma(k) : \mathbb{Z}_+ \mapsto \{0,1\}^M$, satisfying $\|\sigma(k)\|^2 = N, \forall k$.

For example, if $\sigma(k)$, $k = 0, 1, 2, \ldots$, is a $p$-to-$w_\sigma$ communication sequence, then $\sigma_i(k) = 1$ if the $i$-th sensor is to access the controller at time $k$, and $\sigma_i(k) = 0$ otherwise. Similarly, $\rho(k)$ will be a $w_\sigma$-to-$m$ communication sequence prescribing the times at which the actuators are to communicate with the controller. As in [16], at each time $k$, sensors which are not actively communicating will be ignored by the controller; actuators which are not updated at time $k$ will be turned off (zero control). This gives rise to the following dynamics for the deterministic counterpart of (1), with $w_\rho$ inputs and $w_\sigma$ outputs:

$$\begin{aligned} x(k+1) &= Ax(k) + B\mu_\rho(k)^T \bar{u}(k) \\ \bar{y}(k) &= \mu_\sigma(k) C x(k) \end{aligned} \tag{2}$$

where $\bar{u}(k) \in \mathbb{R}^{w_\sigma}$ contains the input data for the actuators which will be granted medium access at $k$, and is produced by the controller based on the available sensors at the time, $\bar{y}(k) \in \mathbb{R}^{w_\rho}$. The notation $\mu_\sigma(k)$ indicates the matrix that results by deleting the $p - w_\sigma$ all-zero rows from the $p \times p$ matrix $\mathrm{diag}(\sigma(k))$.

Equation (2), describes the NCS "from the controller's point of view" and incorporates the dynamics of the plant together with the access status of the communication medium. Perhaps a more practical choice might be to apply a ZOH to the plant's inputs, so that an actuator maintains its level until it receives new data. We will discuss this possibility in the sequel. For now, we will take advantage of the simpler, non-ZOH model in order to facilitate the construction of effective periodic communication sequences. Our approach will also have implications for their suitability when a ZOH *is* in place.

## III. Choosing Effective Communication Sequences

The choice of communication policies $\rho$ and $\sigma$ determines the time-varying dynamics of the NCS (2). Thus, in any optimal control design problem involving (2), we can only say that a controller is optimal for a given communication policy. Optimizing with respect to both control and communication is generally difficult [6]–[8] and often involves combinatorial complexity. Instead, we will relax the requirement for joint optimality and show how to select communication sequences that are "good enough", in the sense that they guarantee the existence of an accompanying optimal controller, and are easy to generate. In particular, we will look for sequences which preserve important structural properties of the underlying LTI plant in (2), including controllability and observability.[1] Such sequences are not unique; however, effective examples can be easily constructed based on the plant's controllability/observability indices, as we will show. We begin by reviewing some relevant terms.

*Definition 2:* The NCS (2) is *controllable on* $[k_0, k_f]$ if, $\forall x_0, \exists \bar{u}(\cdot)$ that steers (2) from $x(k_0) = x_0$ to the origin at time $k_f$. We say that (2) is *l-step controllable*, or simply *controllable* if there exists an integer $l > 0$ such that (2) is controllable on $[k, k+l]$ $\forall k$.

Observability and reconstructibility for the NCS (2) are defined in an analogous manner.

It will be convenient to initially assume that the matrix $A$ in (2) is invertible, so that controllability and reachability of (2) are equivalent, as are observability and reconstructability. The case where $A$ is singular will be treated later in this section. The following theorem from [14] guarantees the existence of periodic controllability (observability)-preserving communication sequences for the NCS under consideration.

*Theorem 1 [14]:* Let the pair $(A, B)$ be controllable, where $B$ is $n \times m$, and $A$ is invertible. For any integer $1 \leq w_\rho < m$, there exist integers $l, N > 0$ and an $N$-periodic $m$-to-$w_\rho$ communication sequence $\rho(\cdot)$, $N \leq \lceil (n/w_\rho) \rceil \cdot n$, such that the extended plant (2) is controllable on $[k, k+l]$ for all $k$, and thus controllable.

The Proof of Theorem 1 includes a sequence construction procedure that results in a controllability-preserving sequence of period at most $N \leq \lceil (n)/(w_\rho) \rceil \cdot n$. The same upper bound is also reported in [17].

### A. Shorter Period Communication Sequences

The sequence construction algorithms given in previous work [15], [16], are known to produce sequences whose period is usually far shorter than the upper bound associated with Theorem 1. The following result shows that in fact $N = n$ steps are sufficient.

*Theorem 2:* Let $(A, B)$ be controllable, where $B$ is $n \times m$, and $A$ is invertible. For any integer $1 \leq w_\rho < m$, there exists an $n$-periodic $m$-to-$w_\rho$ communication sequence $\rho(\cdot)$ such that the extended plant (2) is controllable. In particular, if $n_1, n_2, \ldots, n_m > 0$ are the controllability indices of $(A, B)$, then for $w_\rho = 1$, the sequence whose first period is $\{\underbrace{e_m, e_m \ldots, e_m}_{n_m \text{times}}, \underbrace{e_{m-1}, e_{m-1} \ldots, e_{m-1}}_{n_{m-1} \text{times}}, \ldots, \underbrace{e_1, e_1 \ldots e_1}_{n_1 \text{times}}\}$, where $e_i$ is the $i$-th standard basis vector, makes (2) controllable.

*Proof:* It is sufficient to prove the theorem for the worst-case scenario, $w_\rho = 1$. Assuming zero initial conditions, the matrix

$$R \triangleq \left[B\mu_\rho^T(n-1), AB\mu_\rho^T(n-2), \cdots, A^{n-1}B\mu_\rho^T(0)\right] \tag{3}$$

maps the first $n$ inputs to the state $x(n)$. Of course, (2) is controllable on $[0, n]$ (and thus controllable by [31], [16]), iff $\mathrm{rank}(R) = n$. Each matrix $\mu_\rho^T(k)$ has the effect of "selecting" one of $m$ columns from $A^{n-k-1}B$. If $(A, B)$ is controllable, then we have the following collection of $n$ columns from $R$

$$\begin{aligned} \mathcal{R}_1 \triangleq [&b_1, Ab_1, \ldots, A^{n_1-1}b_1 \\ &b_2, Ab_2, \ldots, A^{n_2-1}b_2, \ldots, b_m, Ab_m, \ldots, A^{n_m-1}b_m] \end{aligned}$$

[1]If we adopt a continuous-time model (e.g., inputs change continuously, but not all simultaneously), then the communication policy selection problem is much simpler and, in fact, "round robin" policies preserve the plant's structural properties [30], as does any periodic policy that devotes some time to every actuator (sensor), in any order [13]. As illustrated here, the situation is not as simple in the discrete-time setting.

is full-rank, where $b_i$ is the $i$th column of $B$, and the positive[2] integers $n_1, \ldots, n_m$ are the controllability indices[3] of $(A, B)$ [33], such that $\sum_1^m n_i = n$, and for $i = 2, \ldots, m$, $n_i$ is the smallest integer such that $A^{n_i} b$ is linearly dependent on the terms $b_1, Ab_1, \ldots, b_i, Ab_i, \ldots, A^{n_i - 1} b_i$ in $\mathcal{R}_1$.

Observe that we can use the last $n_1$ terms of the communication sequence (equivalently, the $\mu_\rho$) to select the first $n_1$ columns of $\mathcal{R}_1$ in $R$, but that after that we are unable to select $b_2$, $Ab_2$ and so on because we are restricted to choosing terms with increasing powers of $A$.

Now, consider the matrix

$$\begin{aligned}\mathcal{R}_2 = [&b_1, Ab_1, \ldots, A^{n_1-1} b_1, A^{n_1} b_2 \\ &A^{n_1+1} b_2, \ldots, A^{n_1+n_2-1} b_2, \ldots \\ &A^{n_1+\ldots+n_{m-1}} b_m, A^{n_1+\ldots+n_{m-1}+1} b_m, \ldots, A^{n-1} b_m].\end{aligned} \tag{4}$$

Clearly, $\mathcal{R}_2$ is obtained from (3) by using the communication sequence suggested in the Theorem's statement. If $\mathrm{rank}(\mathcal{R}_2) = n$, then the theorem is established, because i) it is possible to choose $n$ independent columns, one from each term in $R$, and ii) a sequence that does the job is that which selects the last input for as many steps as the last controllability index of $(A, B)$, the second-to-last input for as many steps as the second-to-last controllability index, and so on.

To show that $\mathrm{rank}(\mathcal{R}_2) = n$, consider first a system with only two inputs, so that $m = 2$ (for $m = 1$ the statement is trivial). Then, $\mathcal{R}_1$ and $\mathcal{R}_2$ specialize to

$$R_1 = [b_1, Ab_1, \ldots, A^{n_1-1} b_1, b_2, Ab_2, \ldots, A^{n_2-1} b_2] \tag{5}$$

and

$$R_2 = [b_1, Ab_1, \ldots, A^{n_1-1} b_1, A^{n_1} b_2, A^{n_1+1} b_2, \ldots, A^{n-1} b_2]. \tag{6}$$

Of course, $\mathrm{rank}(R_1) = n$ by the controllability of $(A, B)$. Because $A$ is invertible, $\mathrm{rank}(A^{n_1} R_1) = n$ as well. Now, notice that the last $n_2$ columns of $R_2$ and $A^{n_1} R_1$ are identical. At the same time, because $n_1$ is a controllability index, we have that the first $n_1$ columns of $A^{n_1} R_1$ are linear combinations of the first $n_1$ columns of $R_2$. Thus, $n = \mathrm{rank}(A^{n_1} R_1) \leq \mathrm{rank}(R_2)$. We conclude that $\mathrm{rank}(R_2) = n$.

For $m = 3$ we have

$$\begin{aligned}R_1 &= [b_1, Ab_1, \ldots, A^{n_1-1} b_1, b_2, Ab_2 \\ &\quad \ldots, A^{n_2-1} b_2, b_3, Ab_3, \ldots, A^{n_3-1} b_3] \\ R_2 &= [b_1, Ab_1, \ldots, A^{n_1-1} b_1, A^{n_1} b_2, A^{n_1+1} b_2, \ldots, \\ &\quad A^{n_1+n_2-1} b_2, A^{n_1+n_2} b_3, A^{n_1+n_2+1} b_3, \ldots, A^{n-1} b_3].\end{aligned}$$

Now, the last $n_3$ columns of $A^{n_1+n_2} R_1$ are identical to those of $R_2$. Also, because $n_1$ is a controllability index, the first $n_1$ columns of $A^{n_1+n_2} R_1$ are linear combinations of the corresponding columns of $R_2$. The remaining $n_2$ columns of $A^{n_1+n_2} R_1$ are linear combinations of the first $n_1 + n_2$ columns of $R_1$, because $n_2$ is a controllability index. Notice that, for the same reason, the first $n_1 + n_2$ columns of $R_2$ are linear combinations of the corresponding columns in $R_1$, and that the converse also holds, because both collections have equal and full rank[4], $n_1 + n_2$. It follows that $A^{n_1+n_2} b_2, \ldots, A^{n_1+2n_2-1} b_2$ are linear combinations of the first $n_1 + n_2$ columns of $R_2$. We conclude that all columns of $A^{n_1+n_2} R_1$ are linear combinations of those of $R_2$, thus $n = \mathrm{rank}(A^{n_1+n_2} R_1) \leq \mathrm{rank}(R_2)$, and $\mathrm{rank}(R_2) = n$ again.

Using successive applications of this argument, we can extend $R_1$, $R_2$ and $R_3$, one controllability index at a time, and show that in each case $\mathrm{rank}(R_2) = n$, thus establishing the theorem. □

From the last theorem, we see that the state dimension, $n$, is a necessary and sufficient number of steps to be able to steer the NCS between any two states. By switching from column to row manipulations, the duality of controllability and observability yields an analogous result for sequences that preserve the observability of the extended plant.

*Remark:* In some cases, such as in LQG control, it suffices to guarantee the weaker properties of stabilizability and detectability in the NCS while removing the assumption of an invertible $A$ in Theorem 2. The following result is obtained by restating [27, Th. 5] (also in [13]) to reflect the existence of period-$n$ communication sequences which preserve the controllability of a pair $(A, B)$, as per Theorem 2.

*Corollary 1:* Let $(A, B)$ be stabilizable, where $B$ is $n \times m$. For any integer $1 \leq w_\rho < m$, there exists an integer $N \leq n$ and an $N$-periodic $m - to - w_\rho$ communication sequence $\rho(\cdot)$ such that the extended plant is stabilizable. A dual corollary holds for detectability.

### B. NCS With an Input ZOH

Our choice of "turning off" actuators when they are not communicating has the effect of avoiding any enlargement of the state vector [4], [5] but may result in inputs with large step changes. A more practical approach is to amend our model to include zero-order holding of inputs [34], [17], by augmenting the state to include the contents of the ZOH, i.e.

$$\begin{aligned}x(k+1) &= Ax(k) + B(I - M_\rho(k)) u_{\mathrm{ZOH}}(k) \\ &\quad + B\mu_\rho^T(k)\bar{u}(k) \\ u_{\mathrm{ZOH}}(k+1) &= (I - M_\rho(k)) u_{\mathrm{ZOH}}(k) + \mu_\rho^T(k)\bar{u}(k) \\ \bar{y}(k) &= \mu_\sigma(k) C x(k).\end{aligned} \tag{7}$$

where $M_\rho(k) \triangleq \mathrm{diag}(\rho(k))$. In the following, we will refer to (7) as the *ZOH-inclusive* model[5]; we will also use $\mathcal{X} \subset \mathbb{R}^{n+m}$ to denote the span of the first $n$ standard basis vectors in $\mathbb{R}^{n+m}$ (i.e., the subspace corresponding to the $x$-states in (7)).

Recent work [17] has proved Th. 1 for the ZOH-inclusive model (7) by retracing the steps of [15]. The next result addresses the question of whether the controllability-preserving sequences obtained using the plant's controllability indices (Theorem 2) remain effective when a ZOH is in place.

*Corollary 2:* Let the pair $(A, B)$ be controllable, where $B$ is $n \times m$, and $A$ is invertible. Let $n_1, \ldots, n_m > 0$ be the controllability indices[6] of $(A, B)$. For $i = 2, \ldots, m$, let $j(i) = \sum_1^{i-1} n_q$, and denote by $\mathcal{C}_i$ the matrix that contains the first $\sum_1^i n_q$ columns of $\mathcal{R}_2$ in (4). Let $V_i \in \mathbb{R}^n$ be the projection of $A^{j(i)} b_i$ onto the nullspace of $\mathcal{C}_i^T$ with row $(A^{j(i)} b_i)^T$ deleted, where $b_i$ is the $i$-th column of $B$. If

$$V_i^T \sum_{q=0}^{j(i)} A^q b_i \neq 0 \; \forall i = 2, \ldots, m-1 \tag{8$$

[2]We assume, without loss of generality, that all $n_i$ are strictly positive, otherwise the system is controllable with fewer than $m$ inputs.

[3]We note that, as defined here, the controllability indices correspond to the lengths of the chains generated by each column of $B$ in the so-called "crate diagram" of the pair $(A, B)$, when that diagram is completed by columns [32]. This is different from the more-frequently used definition, where the crate diagram is completed row-by-row.

[4]Recall that if $V_1, V_2 \in \mathbb{R}^{n \times N}$ with $n > N$, $\mathrm{rank}(V_1) = \mathrm{rank}(V_2) = N$ and $V_1 = V_2 Q$ for some $Q \in \mathbb{R}^{N \times N}$, then $Q$ must be invertible because $N = \mathrm{rank}(V_1) = \mathrm{rank}(V_2 Q) \leq \min(\mathrm{rank}(V_2), \mathrm{rank}(Q)) = \min(N, \mathrm{rank}(Q))$. Thus, $V_2 = V_1 Q^{-1}$, i.e., $V_2$'s columns also are linear combinations of those in $V_1$.

[5]Note that it is unnecessary to apply a ZOH at the sensor side of the NCS (2), because holding a sensor output does not provide any new information to a controller (or observer) with knowledge of the output communication sequence.

[6]Again, we have in mind the definition used in the Proof of Theorem 2.

then there exists an $N$-periodic controllability-preserving communication sequence $\rho$, with $N \leq n$, such that the controllable subspace of (7) contains $\mathcal{X}$ (the span of the first $n$ states in (7))

*Proof:* Let $\rho(\cdot)$ be the $n$-periodic controllability-preserving communication sequence constructed via Th. 2 for the NCS (2) in the worst case, $w_\rho = 1$. We will show that the controllable subspace of (7) under $\rho$ contains $\mathcal{X}$. The state evolution after $n$ steps, with the ZOH in place, is $x(n) = \Lambda \cdot [\bar{u}^T(n-1), \bar{u}^T(n-2), \ldots, \bar{u}^T(0)]^T$, where

$$\Lambda = \Bigg[ B\mu_\rho^T(n-1), AB\mu_\rho^T(n-2) + \sum_{i=0}^{0} A^i B \prod_{j=n-1}^{n-1-i} \bar{M}_\rho(j)\mu_\rho^T(n-2), A^2B\mu_\rho^T(n-3) + \sum_{i=0}^{1} A^i B \prod_{j=n-2}^{n-1-i} \bar{M}_\rho(j)\mu_\rho^T(n-3), \ldots A^{n-2}B\mu_\rho^T(1) + \sum_{i=0}^{n-3} A^i B \prod_{j=2}^{n-1-i} \bar{M}_\rho(j)\mu_\rho^T(1), A^{n-1}B\mu_\rho^T(0) + \sum_{i=0}^{n-2} A^i B \prod_{j=1}^{n-1-i} \bar{M}_\rho(j)\mu_\rho^T(0) \Bigg]$$

$\bar{M}_\rho(k) \triangleq (I - M_\rho(k))$, and we have assumed without loss of generality that $u_{\mathrm{ZOH}}(0) = 0$ in (7). As before, the corollary is established if during a single period of the communication sequence we can choose the $\mu_\rho(k)$ so that $\mathrm{rank}(\Lambda) = n$. Notice that the sequence $\mu_\rho$ selects the same columns from $R$ in (3) as in Theorem 2, but there are now added terms selected at the same time (i.e., the sums in each column of (9)).

Consider the first period of the sequence constructed in Theorem 2, $\rho = \{\underbrace{e_m, e_m \ldots, e_m}_{n_m times}, \ldots, \underbrace{e_1, e_1 \ldots, e_1}_{n_1 times}\}$, so that during the last $n_1$ steps we select the first input. By inspecting the columns of (9) from left to right, we observe that whenever $\rho(n-k) = \rho(n-k-1)$, $k = 1, \ldots, n$, the additional terms in the $(k+1)$st column of $\Lambda$ vanish, because they are multiples of $\bar{M}_\rho(n-k)\mu_\rho^T(n-k-1) = (I - M_\rho(n-k))\mu_\rho^T(n-k) = 0$. The only columns of $\Lambda$ for which the additional terms do not vanish are those which correspond to a change in $\rho(k)$ over the previous step, $k-1$, i.e., those in columns $j(i)+1$, $i = 2, \ldots, m$. All other columns of $\Lambda$ are identical to the corresponding columns of the input-to-state map, $\mathcal{R}_2$ in (4), which is full rank by Theorem 2.

Now, consider constructing $\Lambda$ for $m = 2, 3, \ldots$, and so on. Based on the previous discussion, for $m = 2$ (a two-input plant), the only additional term will be $v = \sum_{i=0}^{n_1-1} A^i b_2$, in the $(n_1+1)$st column—all others will be as in $R_2$ (6). Then, $\Lambda$ is obtained simply by adding $v$ to the $(n_1+1)$st column of $R_2$, $A^{n_1}b_2$, and it is enough to show that doing so does not cause a reduction in rank.

Observe first that the terms $b_2, Ab_2, \ldots, A^{n_1-1}b_2$ summing up to $v$, are linear combinations of the columns of $R_1$ in (5) (if $n_1 \leq n_2$ this is trivial; if $n_1 > n_2$, it follows from $n_2$ being a controllability index). Also, $R_1$ and $R_2$ in (5), (6) are of equal (full) rank, and $R_2$'s columns are linear combinations of those in $R_1$. Using an argument similar to that in the Proof of Theorem 2, we find that $R_1$'s columns (and in particular $b_2, Ab_2, \ldots, A^{n_2-1}b_2$) must also be linear combinations of those in $R_2$ (see footnote 4). We conclude that $v$ is a linear combination of columns already in (6), possibly including $A^{n_1}b_2$ itself. If condition (8) is satisfied, then adding $v$ to $A^{n_1}b_2$ in (6) cannot result in loss of rank, so that $\mathrm{rank}(\Lambda) = rank(R_2) = n$.

Using the same argument, one can show that $\mathrm{rank}(\Lambda) = n$ for any finite $m > 2$. □

The dual statement to Corrollary 2 (concerning observability-preserving sequences used with the ZOH-inclusive system), also holds.

Of course, if (8) is not satisfied, one can still obtain a controllability-preserving communication sequence, using [17]. In that case, the upper bound on the sequence length is $\lceil n/w_\rho \rceil n$, rather than $n$. At this time, there does not seem to be an obvious way of modifying the proof in [17] to reduce the upper bound on the communication sequence length without requiring (8).

## IV. Example: LQG Control

We proceed to illustrate our approach in the case of LQG control of a the stochastic linear NCS

$$\begin{aligned} x(k+1) &= Ax(k) + \bar{B}(k)\bar{u}(k) + v(k) \\ \bar{y}(k) &= \bar{C}(k)x(k) + \bar{w}(k) \end{aligned} \qquad (10)$$

where $\bar{B}(k) \triangleq B\mu_\rho(k)^T$, $\bar{C}(k) \triangleq \mu_\sigma(k)C$, $\bar{w}(k) \triangleq \mu_\sigma(k)w(k)$, and $v(\cdot)$, $w(\cdot)$ and $x(0)$ are as in (1). Note that $\bar{w}(k) \sim N(0, I_{w_\sigma \times w_\sigma})$. Having lifted the requirement for optimal communication, we would like to solve the following problem.

*Problem 1:* Find a pair of periodic input and output sequences $(\rho(\cdot), \sigma(\cdot))$ which preserve the stabilizability and detectability of the NCS (10), and a controller which minimizes

$$J = \mathcal{E}\left\{ \sum_{k=0}^{N_f} x^T(k)Qx(k) + \bar{u}^T(k)\bar{u}(k) \right\}, \quad Q = Q^T. \qquad (11)$$

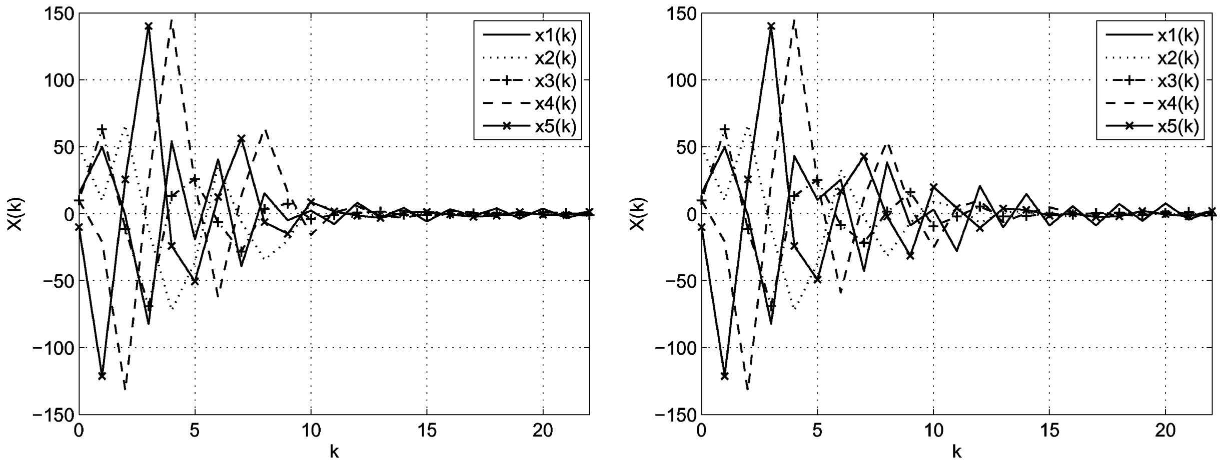
If the communication sequences are constructed as per Section III then Problem 1 becomes a standard LQG design problem for a periodic stochastic plant. A discussion of that problem in the context of NCSs without ZOH can be found in [27]. Briefly, it is well known [35] that the solution consists of a Kalman filter that estimates $x(k)$ from the output $\bar{y}(k)$, and an optimal LQ feedback controller designed under the assumption of perfect state information. From [36], if the stochastic extended plant is stabilizable and detectable (by choice of the sequences $\rho, \sigma$), there exist corresponding optimal periodic Kalman gains, which stabilize the estimation error dynamics, and LQ gains which stabilize the stochastic extended plant (see [27] for a summary of these results in the context of the NCS studied here). Finally, if there are known transmission delays between controller and plant, then a delay compensator of the type discussed in [37] can be combined with the LQG controller and communication sequences developed previously. Details on the construction of the delay compensator in the present context can be found in [27].

We simulated a discrete-time two-input, two-output, fifth-order unstable LTI plant, controlled through a shared communication medium with $w_\rho = w_\sigma = 1$, a two-step plant-to-controller delay, and a three-step controller-to-plant delay. The plant's parameters were

$$A = \begin{bmatrix} -1.05 & 1.05 & -0.5 & 1.05 & -0.775 \\ 0 & 0 & 1.05 & 0 & 0 \\ 0 & 1.05 & 0 & 1.05 & 0 \\ 0 & 0 & -2.1 & 0 & 0 \\ 0 & -2.1 & 1 & -2.1 & 0.5 \end{bmatrix},$$

$$B = \begin{bmatrix} 1 & 0.5 \\ 1 & 0 \\ 1 & 0 \\ -1 & 0 \\ -2 & 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix}.$$




(a) (b)

Fig. 2. (a) State evolution of the closed-loop stochastic NCS (without ZOH) under optimal LQG control. The loop delays were $\Delta_1 = 2, \Delta_2 = 3$. (b) State evolution of the ZOH-inclusive NCS for the same initial conditions and noise sample paths.

This plant is stabilizable and detectable, with a one-dimensional stable controllable/unobservable subsystem, and a four-dimensional unstable controllable/observable subsystem whose corresponding $A$ matrix is singular. The disturbance terms in (1) were $v(\cdot) \sim N(0, 0.35 I_{5\times 5})$ and $w(\cdot) \sim N(0, I_{2\times 2})$. We formulated an infinite-horizon version of the LQG problem, with $Q = 4I_{5\times 5}$ and initial conditions $x(0) = [15, 50, 10, 10, -10]^T$, $\hat{x}(0) = 0$, and $\Sigma(0) = 0.2 \cdot I_{5\times,5}$.

By Corrollary 1, in order to obtain a communication sequence that maintains stabilizability (detectability) in the presence of limited communication, it is sufficient to identify the sequence that does the job for the nonsingular part of the controllable (reconstructible) subsystem. A pair of sequences that accomplishes this (as per Theorem 2) is the three-periodic $\rho(k) = \sigma(k) = \{e_2, e_1, e_1, e_2, e_1, e_1, \ldots\}$ (the controllability and observability indices were both $n_1 = 2, n_2 = 1$). Having thus guaranteed that the NCS is stabilizable and detectable, we applied a delay compensator and optimal LQG controller. It is interesting to note that "round-robin" policies (e.g., $\rho(k) = \{e_1, e_2, e_1, e_2, \cdots\}$, $\sigma(k) = \{e_1, e_2, e_1, e_2, \cdots\}$) did *not* result in a stabilizable or detectable system.

The state evolution of the closed-loop system under optimal control is shown in Fig. 2(a). For comparison, we designed an optimal LQG controller for the ZOH-inclusive periodic NCS (7), with the same cost function penalizing the states and the input (in this case, the contents of the ZOH). The plant satisfied condition (8), thus the communication sequences designed for the non-ZOH NCS preserved the structural properties of the ZOH-inclusive model as well, in the sense of Corrollary 2. The plant parameters, delays, and communication sequences used were the same as in the previous simulation.

The state evolution of the ZOH-inclusive plant is shown in Fig. 2(b). The response was similar to that of the simpler model, but the average LQG cost was in fact higher. In particular, for 1000 runs starting from the same initial conditions and using the same noise sample paths over $k \in [0, 10^4]$, the average cost was $9.124 \times 10^5$ for the non-ZOH model versus $9.654 \times 10^5$ for the ZOH-inclusive NCS. However, in the ZOH-inclusive model the initial transient decayed at a lower cost, $5.407 \times 10^5$, than in its non-ZOH counterpart (10), $5.499 \times 10^5$, taken over the same 1000 runs but for $k \in [0, 15]$. Finally, when the initial states were set to zero, the average cost over 1000 runs with $k \in [0, 500]$ was $1.7862 \times 10^4$ without ZOH, versus $2.076 \times 10^4$ for the ZOH-inclusive model, indicating that the latter is more expensive to control in the long run. In this case, introducing a ZOH did not lead to better long-term performance. We would expect the opposite to be true whenever successive input values are correlated, and have observed such cases in our numerical experiments with various plant parameters. This situation is unusual; one typically expects that a more complex controller will always lead to improved performance (e.g., in [38] and [39] anticipatory control outperforms zero-control and ZOH, respectively).

## V. Conclusion

We discussed the control of NCSs which are subject to medium access constraints and known delays. Because the problem of jointly optimizing the control and communication appears to be intractable, we have opted for separating the two subproblems by restricting communication to periodic sequences which preserve the plant's structural properties under limited communication. We showed how such sequences can be designed for linear NCS in which sensors are ignored by the controller when not granted medium access, while actuators are turned off when they are not receiving data. The sequences developed here were completely determined by the plant's controllability and observability indices.

Eliminating zero-order holding reduces the complexity of the NCS, simplifies the selection of useful communication sequences, and guarantees the least-conservative upper bound on their period (equal to the state dimension). From a practical point of view, a shorter communication period may generally be preferable because it minimizes the maximum time between visits to any sensor/actuator; the controller may thus be able to react to disturbances and/or identify faults sooner compared to when a longer period communication sequence is used. If the plant's parameters satisfy a suitable linear-algebraic condition, then any communication sequences designed for the non-ZOH model, are equally effective in the more practical setting where a ZOH *is* included. Interestingly, the inclusion of a ZOH at the plant's input stage may not always result in a lower LQG cost for long horizon problems. Here we have focused on LQG control, although one could also consider any other control design problem for which there are available results for periodic systems.

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