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Second-Order Accelerator of Investment: The Case of Discrete Time

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Abstract

This paper presents a discrete time version of Hillinger's (1992, 2005) second order accelerator model that investigates the dynamic behavior of capital, for pedagogical purposes. Such a version is put forward as a means of improving student acquaintance with the analysis of investment cycles -defined as quasi-periodic cyclic movements of these variables- and with the convergence towards the steady-state when capital is subjected to trigonometric oscillations. In addition, we extend the analysis, introducing the exogenous interest rate on loans in the behavioral equation of investors. It is inferred that the introduction of this credit term results in a lower equilibrium level of capital.

Keywords: Hillinger's second order accelerator model, investment cycles, educational macroeconomics

JEL Codes: A20, A22, A23, E10, E32, E40

1. Introduction

In order to categorize business cycle theories, different criteria can be used. A common criterion concerns the 'exogeneity' or 'endogeneity' of the cycles. More specifically, if the fluctuations are caused by external shocks, the cycle is 'exogenous'. These shocks may be completely random and non-cyclical. However, most Keynesians follow an alternative approach. They claim that although external shocks may exist and influence the economic activity, fluctuations would occur even

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in the absence of shocks. From this perspective, business cycles are endogenous. The reader can refer to the books of Zarnovitz (1992) or Sushko (2006) to see the history and the theories of business cycles.

The theories that examine the goods' market give major emphasis on the determination of investment (Skott 2012). Taking into consideration that investment constitutes a main part of total expenditure, the existence of high investment results in high aggregate demand and output. Moreover, this high level of output is followed by an increase in capital utilization and profitability that normally leads to an increase in investment and output in the next period. However, the convergence of the economic system towards its long-run equilibrium depends on the sensitivity of investment decisions in capital utilization and profitability. In particular, if this sensitivity is high, the result is a stable time path for output. On the other hand, a low level of sensitivity leads to divergence of output from its long-run equilibrium.

A primal work on business cycles modeling was done by Tinbergen (1930). His model, a model of industrial investment cycle, introduced the concept of time to build. This was achieved assuming the existence of a time lag between the inception and the completion of an investment project in shipping industry. The combination of this discrete time relation with other continuous time relations in this model led to a system of mixed difference-differential equation. Thus, the complexity of the solution of such model may provide an explanation for the limited adherence of this approach.

The argument behind the *General Theory of Employment, Interest and Money* (Keynes 1936) was the role of the Great Depression as the cause of permanent structural changes in industrialized nations' economies. Among the factors of market disequilibrium that Keynes (1936) indicated, the price rigidity is that usually cited in the theory of investment cycles. In addition, the determination of these investment cycles was based on the relationship between the existing capital stock and the demand for output. This was the intuition behind the 'acceleration principle'. Then, the interest shifted to the inventory fluctuations and the inventory cycle, while the accelerator-multiplier mechanism became the centerpiece of Keynesian macroeconomic models. Samuelson (1939) was the first to form an accelerator-multiplier model. His analysis was based on a unit lag in both the consumption and the investment equations. In order to explain the inventory cycle, Metzler (1941) also applied this interaction.

Westerhoff (2006) extended the previous model, introducing a non-linear mix of extrapolative and regressive expectation formation mechanism. That's way the model is useful for the implementation of economic policy as well. Using the Samuelson's analysis as an inception, Jones (2010) showed the usefulness of competition trade theory in the general equilibrium theory. Moreover, Bond & Driskill (2009) studied the concept of dynamic stability in a trade model with three steady-state equilibria. An extension of the Samuelson multiplier-model was also presented by Dassios et al. (2014) who incorporated delayed variables in the initial model. Their analysis achieves to interpret the origination of stable business cycles if realistic and stochastic values of both the multiplier and the accelerator are concerned. Following a similar approach, Dassios & Zimbidis (2014) studied the stability of the national economy in the case of an interaction among different countries. In the same manner, Dassios & Kalogeropoulos (2014) attempted to explain the existence of stable business cycles in the context of a foreign trade model of three countries. In addition, Puu et al. (2005) proposed a Hickian type (Hicks 1950) that concentrates on the "floor" of income and omits the "ceiling". In particular, the "floor" is combined with the depreciation of capital stock. At the same time, they introduced a new dynamic method that is based on the relative growth rates. This model interprets the existence of business cycles with increasing amplitude.

Generally, the theoretical models in this field generate investment cycles and suggest that the roles of consumption and investment are of the same importance. This suggestion is not confirmed by empirical observations which imply that investment is more sensitive to the business cycle than the consumption. Hillinger (1992) developed a model in continuous time, explaining the central role of investment and inventory decision. As we will see, the fact that short run adjustments occur in quantities, not in prices, is a basic assumption of the theory of investment cycles and marks a great divide to the neoclassical paradigm. Hillinger et al. (1992) derived the second order accelerator for both fixed investments and inventories, regarding the microeconomic firm behavior. Hillinger & Weser (1988) and Weser (1992) used the second order accelerator model to discuss the aggregation problem that arises in business cycle theory. In the same manner, Woitek (1995) and Barnett et al. (1996) examined the business cycle stylized facts following an empirical approach.

It is generally accepted that any model is simply a formal representation and approximation designed to capture the essential of some economic process for the

purpose of investigating certain economic behavior and its policy implications. Theoretical models formalize some aspect of economic behavior under a set of assumptions, which are usually chosen so that the model allows an analytical solution. The properties of the model can then be derived and their implications studied. Dynamic models directly incorporate time into their structure. This is usually done in economic modeling by using mathematical systems of difference or differential equations. Our contribution is to develop a version of the SOA (Second Order Acceleration) equation model of Hillinger (1992, 2005) that can be taught to undergraduate and median postgraduate students. Our opinion is that it is more convenient for students to present a model in discrete time given that the initial model is in continuous time (see e.g. Friedberg 2010).

The paper is structured as follows. Firstly, we discuss the theoretical model (section 2). Then, the solution of the model (section 3) and the stability conditions of the system (section 4) are presented. Section 5 shows a graphical representation of the roots. Section 6 introduces the exogenous interest rate on loans into the fundamental behavioral equation of investment. Section 7 concludes this educational paper with a summary of our findings.

2. The Model

We derive the Second Order Accelerator of net investment using the standard flexible accelerator (Hillinger 1992). In discrete time, our model is described by the following set of difference equations:

$$I_t - I_{t-1} = c(I_t^* - I_{t-1}) \quad 0 < c < 1 \quad (1)$$

$$I_t^* = b(K_t^* - K_{t-1}) \quad b > 0 \quad (2)$$

$$I_t = K_t - K_{t-1} \quad (3)$$

where I_t : net investment, I_t^* : the desired level of net investment, K_t : capital, K_t^* : the desired level of capital, c : the speed of adjustment, b : a parameter of investors' behavior.

Equation (1) constitutes the partial adjustment mechanism for net investments. It shows that the net investment is adjusted towards the desired level of investment (I_t^*) partially. The coefficient $c \in (0,1)$ is the speed of adjustment. The closer to the unity is the value of c , the faster is the adjustment of net investment in the present period. In

extreme cases, where $c = 0$ and $c = 1$, there is no adjustment and there is full adjustment respectively. The last two cases are not applied to our model. The main intuition behind the partial adjustment is the existence of adjustment costs. More specifically, if firms do not face adjustment costs ($c = 1$), net investment (I_t) adjusts fully to its desired level (I_t^*), that is $I_t = I_t^*$. In this model, the presence of adjustment costs results in the partial adjustment of net investment (I_t) towards its desired level (I_t^*), which means that $I_t = cI_t^* + (1 - c)I_{t-1}$.

Relation (2) is a behavioral equation. It expresses the desired level of net investment (I_t^*) as a fraction of the difference between the desired level of capital (K_t^*) and the actual level of capital with a one-period lag (K_{t-1}). The constant b is positive, reflecting the positive relation between net investment (I_t^*) and the aforementioned difference.

It is noteworthy that the introduction of the time pattern of the investment expenditure leads to a time lag in the transformation of this expenditure into capital. In particular, we follow the time-to-build and time-to-plan approaches according to which the costs of investment projects are incurred with time lags and become productive only when the project is complete. It should be clear that the existence of adjustment lags is in no sense an indication of irrational behavior. There are costs incurred if the various lags are shortened and other costs if they are lengthened. Under the assumption of a finite time path, we presume that the desired level of capital is stable. This allows the notation of the desired level of capital with K^* for the rest of our analysis.

Finally, equation (3) is the definition of net investment. Net investment (I_t) is defined as the change in stock of capital ($K_t - K_{t-1}$).

Taking into consideration that the determination of investment cycles is an eminently dynamic process, our analysis can be induced into the sphere of physical dynamic phenomena (Hillinger (2005)). In particular, the conceptualizations of inertia, kinetic and potential energy can apply from physics to our model. To begin with inertia, it is defined as the property of objects' to resist in any change in their state of motion. In fact, this is an important characteristic of investment that can be realized via the time-lag between the investment plan and its denaturation into capital. Moreover, kinetic energy in physics, i.e. the energy of an object due to its motion, can be paralleled with the investment flow in our system. Similarly, the analogous of

potential energy, the energy due to the object's position, is the capital stock. The key of such an approach is that investment's motivation is the increase in the stock of capital. The existence of excess capital (potential energy) results in the reduction of investment, which generates investment flows (kinetic energy).

Figure 1: Phase Diagram

The comparison of the dynamic terminology of physics with the dynamic properties of our model can be comprehensible by Figure 1. The figure is drawn for the case of a constant amplitude cycle. Capital is shown on the horizontal axis while net investment is presented in the vertical axis. Both variables are measured in terms of deviation from equilibrium. $\bar{I}$ and $\bar{K}$ denote the equilibrium level of net investment and capital respectively. In the first quadrant, the deviation of capital from its equilibrium is positive so there is excess capital. This excess capital leads to a decrease in the investment's deviation from its equilibrium, which is positive, to zero

when excess capital is at maximum. After reaching its maximum, the latter declines resulting in the continuation of the deceleration of investment, which is now, negative (disinvestment). This is the case in quadrant II. When capital reaches its equilibrium level, disinvestment is maximal. In the third quadrant, there is a capital deficit. As capital decreases more and more under its equilibrium level, the investment increases remaining negative. When the capital deficit is at maximum, the disinvestment ceases. In quadrant IV, the capital deficit reduces gradually causing an increase in investment. The latter is positive and reaches at a maximum when the capital deficit is eliminated. Then, the process is repeated. Hence, it is obvious the role of capital as the potential energy of the system and the function of investment as the kinetic energy.

3. Solution of the Model

Inserting a time lag in equation (3), we get:

$$I_t = K_t - K_{t-1} \Rightarrow$$

$$I_{t-1} = K_{t-1} - K_{t-2} \quad (4)$$

The reduced form of our structural model is obtained after the substitution of relations (2), (3) & (4) into (1) as follows:

$$I_t - I_{t-1} = c(I_t^* - I_{t-1}) \Rightarrow$$

$$K_t - K_{t-1} - (K_{t-1} - K_{t-2}) = c[b(K^* - K_{t-1}) - (K_{t-1} - K_{t-2})] \Rightarrow$$

$$K_t - K_{t-1} - K_{t-1} + K_{t-2} = cbK^* - cbK_{t-1} - cK_{t-1} + cK_{t-2} \Rightarrow$$

$$K_t - 2K_{t-1} + K_{t-2} + cbK_{t-1} + cK_{t-1} - cK_{t-2} = cbK^* \Rightarrow$$

$$K_t + (cb + c - 2)K_{t-1} + (1 - c)K_{t-2} = cbK^* \Rightarrow$$

$$K_t + [c(b + 1) - 2]K_{t-1} + (1 - c)K_{t-2} = cbK^* \quad (5)$$

Relation (5) is a second-order difference equation with constant coefficients. In other words, the above relation is the second order accelerator. The general solution of this non-homogeneous equation is derived adding the general solution of the corresponding homogeneous equation and any particular solution of the non-homogeneous equation (5). To begin with the general solution of the homogeneous equation, it shows the deviation of capital from equilibrium.

The homogeneous equation, as this is obtained by (5), has the following mathematical form:

$$K_t + [c(b+1) - 2]K_{t-1} + (1-c)K_{t-2} = 0 \quad (6)$$

Consider that λ^t is a solution of equation (6). Then, we have:

$$\lambda^t + [c(b+1) - 2]\lambda^{t-1} + (1-c)\lambda^{t-2} = 0 \Rightarrow$$

$$\lambda^{t-2}[\lambda^2 + [c(b+1) - 2]\lambda + (1-c)] = 0 \Rightarrow$$

$$\lambda^2 + [c(b+1) - 2]\lambda + (1-c) = 0 \quad (7)$$

Equation (7) is called characteristic equation of the homogeneous equation (6). In order to solve this second order equation, we calculate the discriminant:

$$\begin{aligned} \Delta &= [c(b+1) - 2]^2 - 4(1-c) \\ &= c^2(b+1)^2 - 4c(b+1) + 4 - 4 + 4c \\ &= c^2(b+1)^2 - 4cb - 4c + 4c \\ &= c^2(b+1)^2 - 4cb \quad (8) \end{aligned}$$

Three cases are possible:

1. If the discriminant is positive ($\Delta > 0$), the characteristic equation has two real and distinct roots. These are:

$$\lambda_{1,2} = \frac{-[c(b+1) - 2] \pm \sqrt{c^2(b+1)^2 - 4cb}}{2}$$

The general solution of the homogeneous equation (6) is:

$$\begin{aligned} K_t &= A_1\lambda_1^t + A_2\lambda_2^t \Rightarrow \\ K_t &= A_1 \left[\frac{-[c(b+1)-2] + \sqrt{c^2(b+1)^2 - 4cb}}{2} \right]^t + A_2 \left[\frac{-[c(b+1)-2] - \sqrt{c^2(b+1)^2 - 4cb}}{2} \right]^t \quad (9) \end{aligned}$$

where $A_1, A_2 \in \mathbb{R}$ are arbitrary constants which can be derived using two initial conditions.

According to Vieta's formulas, if x_1 and x_2 are roots of the equation $ax^2 + \beta x + \gamma = 0, a \neq 0$, then it holds:

$$x_1 + x_2 = -\frac{\beta}{a} \text{ \& } x_1 \cdot x_2 = \frac{\gamma}{a}.$$

The roots λ_1 & λ_2 should satisfy Vieta's formulas, that is:

$$\lambda_1 + \lambda_2 = -\frac{[c(b+1) - 2]}{1} = -[c(b+1) - 2]$$

and

$$\lambda_1 \cdot \lambda_2 = \frac{1-c}{1} = 1-c > 0.$$

From the last relation it is inferred that the roots λ_1 & λ_2 are of the same sign. Particularly, in the case of $\lambda_1 + \lambda_2 = -[c(b+1) - 2] > 0$, λ_1 & λ_2 are positive. This implies a monotonic movement of capital. On the other hand, if $\lambda_1 + \lambda_2 = -[c(b+1) - 2] < 0$, these roots are negative and capital shows improper oscillations. In any case, capital (K_t) converges towards its long-run equilibrium if, and only if, both roots are less the unity in absolute values ($|\lambda_1|, |\lambda_2| < 1 \Rightarrow -1 < \lambda_1 < 1$ & $-1 < \lambda_2 < 1$). Otherwise, capital diverges from its equilibrium.

2. If the discriminant is null ($\Delta = 0$), the characteristic equation has a multiple root with multiplicity two ($\alpha(\lambda_0) = 2$):

$$\lambda_0 = \frac{-[c(b+1) - 2]}{2}$$

The general solution of the homogeneous equation (6) is:

$$K_t = A_1 \lambda_0^t + A_2 t \lambda_0^t \Rightarrow$$

$$K_t = (A_1 + A_2 t) \lambda_0^t$$

$$K_t = (A_1 + A_2 t) \left[\frac{-[c(b+1) - 2]}{2} \right]^t, A_1, A_2 \in \mathbb{R} \text{ (10)}$$

where $A_1, A_2 \in \mathbb{R}$ are arbitrary constants which can be derived using two initial conditions.

In this case, capital (K_t) converges towards its long-run equilibrium if, and only if, the absolute value of the root λ_0 is less than unity ($|\lambda_0| < 1 \Rightarrow -1 < \lambda_0 < 1$). On the contrary, if λ_0 is greater than one in absolute values, capital diverges from its steady-state. Regarding the kind of this movement, the convergence will be monotonic if the value interval of λ_0 is the (0,1), while it will be (improper) oscillatory if the value interval is (-1,0). Similarly, the divergence will be monotonic if the value interval of λ_0 is the (1,+ ∞), while it will be oscillatory if the value interval is (- ∞ ,-1).

3. Consider the discriminant is negative ($\Delta < 0$). The roots of the characteristic equation (7) are two complex conjugate numbers. Their mathematical form has as follows:

$$\lambda_{1,2} = \frac{-[c(b+1)-2] \pm i\sqrt{|c^2(b+1)^2 - 4cb|}}{2} \Rightarrow$$

$$\lambda_{1,2} = -\frac{c(b+1)-2}{2} \pm \frac{\sqrt{4cb - c^2(b+1)^2}}{2} i$$

where $i = \sqrt{-1}$ is the imaginary unit.

The general solution of the homogeneous difference equation is given by:

$$K_t = r^t [A_1 \cos \omega t + A_2 \sin \omega t], A_1, A_2 \in \mathbb{R} \quad (11)$$

where $A_1, A_2 \in \mathbb{R}$ are arbitrary constants which can be derived using two initial conditions

In this case, capital (K_t) shows trigonometric oscillations with period equal to $2\pi/\omega$ and amplitude dependent on the modulus or absolute value (r) of the complex conjugate roots. This modulus (r) is obtained denoting first the roots with $\lambda_{1,2} = h \pm ui, i = \sqrt{-1}$:

$$r = \sqrt{h^2 + u^2} = \sqrt{\left[\frac{-[c(b+1)-2]}{2}\right]^2 + \left[\frac{\sqrt{4cb - c^2(b+1)^2}}{2}\right]^2}$$

$$\begin{aligned}
&= \sqrt{\frac{c^2(b+1)^2 - 4c(b+1) + 4 + 4cb - c^2(b+1)^2}{4}} \\
&= \sqrt{\frac{c^2(b+1)^2 - 4cb - 4c + 4 + 4cb - c^2(b+1)^2}{4}} \\
&= \sqrt{\frac{4(1-c)}{4}} \\
&= \sqrt{1-c}
\end{aligned}$$

where $h = -\frac{c(b+1) - 2}{2}$ & $u = \frac{\sqrt{4cb - c^2(b+1)^2}}{2}$

From the assumptions of our structural model concerning the value intervals of the parameters, it is deduced that the absolute value of the complex conjugate roots is less than one². Consequently, the trigonometric oscillations of capital have decreasing amplitude. The value of the modulus r determines the existence of convergence (divergence) of capital (K_t) towards its steady-state. Taking into consideration that $r = \sqrt{1-c} < 1$, capital converges towards its steady-state.

We move now to the particular solution of the non-homogeneous equation (5) using the general method of undetermined coefficients.

Let $\bar{K} = \Gamma, \Gamma \in \mathbb{R}$ a particular solution of (5). Substitution in (5) gives:

$$\Gamma + [c(b+1) - 2]\Gamma + (1-c)\Gamma = cbK^* \Rightarrow$$

$$[1 + c(b+1) - 2 + (1-c)]\Gamma = cbK^* \Rightarrow$$

$$[1 + cb + c - 2 + 1 - c]\Gamma = cbK^* \Rightarrow$$

$$cb\Gamma = cbK^* \xrightarrow{cb>0}$$

$$\Gamma = K^* \text{ (12)}$$

² Proof $r < 1$:

From relation (1), we get:

$$0 < c < 1 \Rightarrow c > 0 \Rightarrow -c < 0 \Rightarrow 1 - c < 1 \xrightarrow{y=\sqrt{x} \uparrow} \sqrt{1-c} < \sqrt{1} \Rightarrow \sqrt{1-c} < 1 \quad \text{O.E.D.}$$

Thus, the particular solution of the non-homogeneous equation is $\bar{K} = \Gamma = K^*$. This solution is interpreted as the equilibrium of capital, so it should be positive. The result is that the steady-state of capital is equal with capital's desired level.

In conclusion, it has to be mentioned that our interest is in the case of the trigonometric oscillatory movement of capital. For this reason we concentrate on the possibility of negative discriminant ($\Delta < 0$). Hence, the general solution of the non-homogeneous difference equation (5) is given by the sum:

$$K_t = r^t[A_1 \cos \omega t + A_2 \sin \omega t] + K^* \quad (13)$$

where $A_1, A_2 \in \mathbb{R}$ are arbitrary constants which can be derived using two initial conditions.

At this point, it is necessary to mention that damped business cycles are triggered by random shocks. As Hillinger (2005, p.34) quotes:

The answer is contained in Ragnar Frisch's classic 1933 article: Propagation problems and impulse problems in dynamic economics. He gives the example of a pendulum whose endogenous behavior is described by a damped periodic differential equation. Subject to random shocks, the pendulum will exhibit an erratic, but still periodic motion; a quasi-cycle in the terminology of the present paper. Similarly, as long as the endogenous adjustment mechanisms of fixed and inventory investment exhibit complex roots, disturbances of any kind can produce the observed quasi-cyclical behavior.

4. Stability Conditions

In order to examine if the behavior of capital is stable over time, the so-called stability conditions are applied to our economic model. According to Gandolfo (1996), the necessary and sufficient stability conditions for a second order difference equation with functional form $\lambda^2 + \alpha_1 \lambda + \alpha_2 = 0$ are:

$$1 + \alpha_1 + \alpha_2 > 0$$

$$1 - \alpha_2 > 0$$

$$1 - \alpha_1 + \alpha_2 > 0$$

In this present paper, where the characteristic equation is given by (7):

$$\lambda^2 + [c(b + 1) - 2]\lambda + (1 - c) = 0,$$

the inequalities that ensure stability are described as follows:

$$\begin{cases} 1 + [c(b+1) - 2] + (1-c) > 0 \Rightarrow 1 + cb + c - 2 + 1 - c > 0 \Rightarrow cb > 0 \\ 1 - (1-c) > 0 \Rightarrow 1 - 1 + c > 0 \Rightarrow c > 0 \\ 1 - [c(b+1) - 2] + (1-c) > 0 \Rightarrow 4 - c(b+2) > 0 \Rightarrow c(b+2) < 4 \end{cases} \quad (14)$$

Both the first and the second stability conditions are satisfied as $0 < c < 1$ & $b > 0$. While this is true, the last stability condition (relation (14)) requires $c(b+2) < 4$. Therefore, inequality (14) is the crucial condition for the existence of dynamic stability.

5. Graphical Presentation of the Roots

In this section, we attempt to show graphically all the possible cases of the characteristic equation's roots and stability. For this reason the discriminant (8) and the crucial stability condition (14) are used. In particular, we depict the functions:

$$c = \frac{4b}{(b+1)^2}, b \neq -1 \quad (15)$$

and

$$c = \frac{4}{b+2}, b \neq -2 \quad (16)$$

The following graph illustrates all the possible cases that were discussed in the previous sections.

Figure 2: Coefficient of investors' behavior-speed of adjustment diagram

The values of the parameter b and the values of the speed of adjustment c are shown on the horizontal and vertical axes respectively. The space of the combinations (b, c) is divided into four segments, A, B, C and D. Taking into consideration that $0 < c < 1$ & $b > 0$, these segments include only the points of the first quadrant that lie below the line $c = 1$.

To begin with region A, these points lie above the function $c = 4b/(b + 1)^2$ so it holds $\Delta > 0$. This implies that the characteristic equation has two real roots. Furthermore, they lie below the function $c = 4/(b + 2)$ so $c(b + 2) < 4$. Therefore, capital converges towards its steady-state. In addition, the points of section B lie below both the functions (15) & (16). In other words $\Delta < 0$ and $c(b + 2) < 4$. The roots are complex and the crucial stability condition is satisfied. The result is a trigonometric oscillatory movement with convergence towards the steady-state.

Regarding section C, the corresponding points lie above the function $c = 4b/(b + 1)^2$, that is $\Delta > 0$, but below the function $c = 4/(b + 2)$. Consequently, the roots of the characteristic equation are real and the crucial stability condition is satisfied. Finally, the points of segment D lie above both the functions (15) & (16). Hence, $\Delta > 0$ and $c(b + 2) > 4$. The result is real roots of the characteristic equation and divergence from the long-run equilibrium.

The previous analysis demonstrates that when the roots of the characteristic equation are complex, the crucial stability condition is satisfied. Consequently, in the case of trigonometric oscillatory movement, capital converges towards the systems' steady-state. While this is true, when the roots are real, there is also the possibility of no satisfaction of (14) and thus divergence from the equilibrium over time.

To conclude, it is necessary to refer to the points that lie on the boundary lines demarcating the four segments. The points that lie on the curve $c = 4b/(b + 1)^2$ satisfy the equation $\Delta = 0$ and as a result the characteristic equation has a real root with multiplicity two. Moreover, it holds that $c(b + 2) < 4$ so the system is stable ($|\lambda_0| < 1$). On the other hand, the points of the curve $c = 4/(b + 2)$ lie above the curve $c = 4b/(b + 1)^2$, i.e. $\Delta > 0$. The path of the capital just separates stability from instability.

6. Extension of the Model: The Exogenous Credit Term

We extend our model, introducing an exogenous credit term, the interest rate on loans. Our structural model is changed as follows:

$$I_t - I_{t-1} = c(I_t^* - I_{t-1}) \quad 0 < c < 1 \quad (17)$$

$$I_t^* = b(K_t^* - K_{t-1}) + qr_L \quad b > 0, q < 0 \quad (18)$$

$$I_t = K_t - K_{t-1} \quad (19)$$

where c : the speed of adjustment, b : a parameter of investors' behavior, q : a parameter that shows the negative relation between the interest rate on loans and the desired level of net investments, r_L : the exogenous interest rate on loans

Relations (17) and (19) are the same with the relations (1) and (2) in our initial model. They describe the partial adjustment mechanism and the definition of net investment respectively. Equation (18) constitutes the behavioral equation of investors and describes the formulation of the desired level of net investment (I_t^*). The difference from our initial model lies in the introduction of a monetary term, the interest rate on loans. It is assumed that this interest rate is given as it is determined exogenously in the credit market. Finally, the desired level of capital is presumed stable over time as before. This is the reason why it is denoted with K^* in the rest of this section.

Following the same methodology as in the initial model, we begin the solution of this extended model with the introduction of a time lag in equation (19).

$$I_t = K_t - K_{t-1} \Rightarrow$$

$$I_{t-1} = K_{t-1} - K_{t-2} \quad (20)$$

The substitution of the relations (18) ~ (20) in (17) leads to the reduced form of the present model.

$$I_t - I_{t-1} = c(I_t^* - I_{t-1}) \Rightarrow$$

$$K_t - K_{t-1} - (K_{t-1} - K_{t-2}) = c[b(K^* - K_{t-1}) + qr_L - (K_{t-1} - K_{t-2})] \Rightarrow$$

$$K_t - K_{t-1} - K_{t-1} + K_{t-2} = cbK^* - cbK_{t-1} + cqr_L - cK_{t-1} + cK_{t-2} \Rightarrow$$

$$K_t - 2K_{t-1} + K_{t-2} + cbK_{t-1} + cK_{t-1} - cK_{t-2} = cbK^* + cqr_L \Rightarrow$$

$$K_t + (cb + c - 2)K_{t-1} + (1 - c)K_{t-2} = cbK^* + cqr_L \Rightarrow$$

$$K_t + [c(b + 1) - 2]K_{t-1} + (1 - c)K_{t-2} = cbK^* + cqr_L \textbf{(21)}$$

Relation (21) is a second-order difference equation with constant coefficients. The solution of this equation requires the determination of the general solution of the corresponding homogeneous difference equation and the derivation of a particular solution of the non-homogeneous equation (21).

The homogeneous equation that is implied by (21) is:

$$K_t + [c(b + 1) - 2]K_{t-1} + (1 - c)K_{t-2} = 0 \textbf{(22)}$$

Comparing the homogeneous-equation (6) with the homogeneous equation (22), we infer that they are exactly the same. Therefore, the introduction of the exogenous interest rate on loans does not affect the deviation of the system from its steady-state. In other words, the general solution of the homogeneous equation is the same as before. Moreover, due to the fact that the stability conditions are based on the homogeneous equation, they also remain unaffected by the credit term. What is influenced by the interest rate on loans is the particular solution of the equation (5), i.e. the equilibrium level of capital.

Applying the general method of undetermined coefficients, we attempt to find a particular solution of (21). Let $\bar{K} = \Gamma$, $\Gamma \in \mathbb{R}$, a particular solution of (21). After the substitution in (21), we get:

$$\Gamma + [c(b + 1) - 2]\Gamma + (1 - c)\Gamma = cbK^* + cqr_L \Rightarrow$$

$$[1 + c(b + 1) - 2 + (1 - c)]\Gamma = cbK^* + cqr_L \Rightarrow$$

$$[1 + cb + c - 2 + 1 - c]\Gamma = cbK^* + cqr_L \Rightarrow$$

$$cb\Gamma = cbK^* + cqr_L \xrightarrow{cb>0}$$

$$\Gamma = \frac{cbK^* + cqr_L}{cb} \Rightarrow$$

$$\Gamma = K^* + (q/b)r_L, \quad b \neq 0 \text{ (which holds)} \textbf{(23)}$$

Hence, the particular solution of the non-homogeneous equation is $\bar{K} = \Gamma = K^* + (q/b)r_L$. This solution can be interpreted as the equilibrium level of capital only if it is positive, that is $K^* + (q/b)r_L > 0 \Rightarrow K^* > -(q/b)r_L$. The comparison of this solution with the corresponding particular solution of (5) leads to the following ascertainment; the introduction of the exogenous interest rate on loans in the initial second order accelerator model in discrete time results in the reduction of the equilibrium level of capital below its desired level. To demonstrate, it holds $K^* + (q/b)r_L < K^*$, as the parameter q is negative while the parameter b is positive.

Concentrating in the case of trigonometric oscillations of capital as before, the general solution of the non-homogeneous difference equation (21) is given by:

$$K_t = r^t[A_1 \cos \omega t + A_2 \sin \omega t] + K^* + (q/b)r_L \quad (24)$$

7. Conclusion

In this paper, we built a second order accelerator model in discrete time. Beginning with a three-equation structural model and making proper substitutions, we derived the reduced form of this model. The latter was described by a second order difference equation with constant coefficients. The solution of this equation yielded the behavior over time of capital. Our interest was in the case of cyclical behavior of capital. Moreover, it was demonstrated that the equilibrium level of capital is equal to its desired level. The graphical representation of all the possible cases occurred showed that when the trigonometric oscillation is the case, capital converges towards its steady-state.

Finally, we proved that the introduction of the exogenous interest rate on loans in the behavioral equation of our structural model affects neither the deviation of capital from equilibrium nor the period of the oscillation in the case of trigonometric oscillatory movement. Additionally, the possibilities of convergence towards the steady-state are not affected as the stability conditions are exactly the same. The result is that the influence of the exogenous credit term lies in the reduction of the equilibrium level of capital below its desired level.

This paper prompts several important issues as subjects of future research. Firstly, it would be very interesting to examine the ability of our model to interpret the existence of investment cycles when an endogenous interest rate on loans is

incorporated in the initial three-equation model. Such an approach would allow us to examine the effects of monetary policy as well. Given capital movements, one might also incorporate into the model the public sector in order to investigate the influence of this sector on the period of the business cycle. Finally, an empirical methodology could be followed to investigate if our model is able to interpret the stylized facts of real economy for different countries.

To conclude, we have to answer what the implication of the existence of investment cycles in this context is. As Hillinger (2005, p.33) writes:

“The most direct implication is a strengthening of the Friedman-neoclassical position against an active anti-cyclical policy, even though the argument leading there is quite different. Monetary and fiscal policies may well be effective; the problem is that given the endogenous dynamics, there is no simple way of calculating what the actual effect will be. For example, an attempt of inflating a currently sluggish economy may only contribute to the excess of an investment boom that was already in the making.”

8. Appendix

In this section, we present analytically the investigation of the functions (15) and (16) that is necessary for plotting Figure 2.

Firstly, we set the discriminant (8) equal to zero.

$$\begin{aligned}\Delta &= c^2(b+1)^2 - 4cb = 0 \Rightarrow \\ c^2(b+1)^2 &= 4cb \xrightarrow{c>0} \\ c &= f(b) = \frac{4b}{(b+1)^2}, b \neq -1 \quad (15)\end{aligned}$$

The domain of function (15) is the interval $A=(0, +\infty)$ as b is assumed positive. The determination of the monotony of (15) requires the investigation of the sign of the first derivative of (15). For this reason, we calculate the latter:

$$\begin{aligned}\frac{\partial c}{\partial b} &= \frac{4(b+1)^2 - 4 \cdot 2b(b+1)}{(b+1)^4} \\ &= \frac{4(b^2 + 2b + 1) - 8b^2 - 8b}{(b+1)^4} \\ &= \frac{4b^2 + 8b + 4 - 8b^2 - 8b}{(b+1)^4}\end{aligned}$$

$$\begin{aligned}
&= \frac{-4b^2 + 4}{(b + 1)^4} \\
&= \frac{-4(b + 1)(b - 1)}{(b + 1)^4} \\
&= -\frac{4(b - 1)}{(b + 1)^3} \quad (\mathbf{A.1})
\end{aligned}$$

The derivative (A.1) is defined for $b \neq -1$ which holds as $b > 0$.

The next step is to set (A.1) equal to zero in order to find the stationary points.

$$\frac{\partial c}{\partial b} = 0 \Rightarrow -\frac{4(b - 1)}{(b + 1)^3} = 0 \Rightarrow b = 1$$

The value $b = 1$ is the critical value of (A.1) because $f'(1) = 0$. The following table summarizes the signs of (A.1) and the monotony of (15) for the different intervals of the latter's domain.

b	$(0,1)$	1	$(1,+\infty)$
$-4(b-1)$	+	0	-
$(b + 1)^3$	+	+	+
$\frac{\partial c}{\partial b}$	+	0	-
$c = \frac{4b}{(b + 1)^2}$		Local maximum

Table A.1: Determination of the monotony of function (15)

It is obvious that function (15) is strictly increasing in the interval $(0, 1)$ while it is strictly decreasing in the interval $(1, +\infty)$. According to the first derivative criterion, the value of this function at $b=1$, $f(b)=f(1)=1$, is a local maximum as the derivative $f'(b)$ changes its sign from positive to negative from immediate left to $b=1$ to its immediate right (Chiang 1984).

We continue with the determination of the curvature of (15) in the different intervals of its domain. For this reason, we find the second derivative of (15).

$$\begin{aligned}
\frac{\partial^2 c}{\partial b^2} &= \frac{-4(b+1)^3 + 4(b-1) \cdot 3(b+1)^2}{(b+1)^6} \\
&= \frac{-4(b^3 + 3b^2 + 3b + 1) + 12(b-1)(b^2 + 2b + 1)}{(b+1)^6} \\
&= \frac{-4b^3 - 12b^2 - 12b - 4 + 12b^3 + 24b^2 + 12b - 12b^2 - 24b - 12}{(b+1)^6} \\
&= \frac{8b^3 - 24b - 16}{(b+1)^6} \\
&= \frac{8(b^3 - 3b - 2)}{(b+1)^6} \quad (\text{A.2})
\end{aligned}$$

The derivative (A.2) is defined for $b \neq -1$ which holds as $b > 0$.

Setting (A.2) equal to zero, we obtain any possible inflection point of (15).

$$\begin{aligned}
\frac{\partial^2 c}{\partial b^2} = 0 &\Rightarrow \frac{8(b^3 - 3b - 2)}{(b+1)^6} = 0 \Rightarrow \\
8(b^3 - 3b - 2) &= 0 \Rightarrow \\
b^3 - 3b - 2 &= 0 \Rightarrow \\
(b+1)(b^2 - b - 2) &= 0 \Rightarrow \\
b = 2 \text{ και } b = -1 &\Leftrightarrow a(-1) = 2 \text{ (rejected)}
\end{aligned}$$

The second derivative (A.2) changes its sign flanking the point $(2, f(2))$, which means that function (15) has an inflection point. More specifically, (15) is concave in the interval $(0, 2)$ and convex in the interval $(2, +\infty)$.

b	(0,2)	2	(2, +∞)
$8(b+1)$	+	+	+
$b^2 - b - 2$	-	0	+
$(b+1)^6$	+	+	+
$\frac{\partial^2 c}{\partial b^2}$	-	0	+
$c = \frac{4b}{(b+1)^2}$		Inflection point

Table A.2: Determination of the curvature of function (15)

Table A.2 presents the sign of (A.2) and the curvature of (15) in the different intervals of its domain.

The overall received results are depicted in the next table:

b	(0,1)	1	(1,2)	2	(2,+∞)
$\frac{\partial^2 c}{\partial b^2}$	-	-	-	0	+
$\frac{\partial c}{\partial b}$	+	0	-	-	-
$c = \frac{4b}{(b+1)^2}$		Local maximum		Inflection point

Table A.3: Investigation of function (15)

Now, we move to the investigation of function (16). Starting with setting the crucial stability condition (14) equal to zero, we get relation (16):

$$c(b+2) = 4 \Rightarrow c = \frac{4}{b+2}, b \neq -2 \quad (\mathbf{16})$$

The domain of (16) is the interval $A=(0, +\infty)$ as b is assumed positive.

The first derivative of (16) is calculated as follows:

$$\frac{\partial c}{\partial b} = -\frac{4}{(b+2)^2} \quad (\mathbf{A.3})$$

The derivative (A.3) is defined only for $b \neq -2$ which holds since $b > 0$. It is negative for any value of b , that is the function (16) is strictly decreasing all over its domain.

The second derivative of (16) is given by:

$$\frac{\partial^2 c}{\partial b^2} = \frac{8}{(b+2)^3} \quad (\mathbf{A.4})$$

As relation (A.3), the derivative (A.4) is defined only for $b \neq -2$ which holds since $b > 0$. It is positive for any value of b , that is the function (16) is strictly convex all over its domain.

The overall received results of the investigation of (16) are summarized in Table A.4:

b	$(0, +\infty)$
$\frac{\partial^2 c}{\partial b^2}$	$+$
$\frac{\partial c}{\partial b}$	$-$
$c = \frac{4b}{(b+1)^2}$

Table A.4: Investigation of function (16)

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