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The Elasticity of Substitution as an Engine of Growth*

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Abstract

This paper characterizes the elasticity of factor substitution in one-sector convex growth models with a general production function. It shows that an elasticity of substitution that is asymptotically greater than one is a sufficient (but not a necessary) condition for the existence of a lower bound on the marginal product of capital, which in turn can lead to unbounded endogenous growth. Hence, an elasticity of substitution that becomes eventually greater than one can counteract the role of diminishing returns to capital. This renders factor substitution a powerful engine of growth.

Keywords: Elasticity of Substitution, Endogenous Growth, Convex Models

JEL Classification: O40; O11

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1. Introduction

In a well-known paper Jones and Manuelli (1990) have shown that if the market interest rate, which in a competitive economy equals the marginal product of capital, is bounded away from zero, then there can be endogenous growth even in the case where there exist diminishing returns with respect to factors that can be accumulated (see also Jones and Manuelli 1997).

On the other hand, starting with Solow (1956), a seemingly unrelated literature has analyzed the role played by the elasticity of substitution in the growth process. Specifically, using a *constant* elasticity of substitution (CES) production function this literature has shown that an elasticity of substitution between capital and labor greater than one makes the emergence of endogenous growth possible (see, for example, La Grandville 1989, Klump and La Grandville 2000, Duffy and Papageorgiou 2000, and Klump and Preissler 2000). Moreover, a higher elasticity of substitution can lead to higher long-term growth rates. Thus, "when two countries start from common initial conditions, the one with the higher elasticity of substitution will always experience, other things being equal, a higher income per head" (Klump and La Grandville 2000, p. 283).

This paper investigates more closely the connection between these two strands of the growth literature and establishes a relationship between the existence of a lower bound on the marginal product of capital and the asymptotic value of the elasticity of substitution in one-sector convex growth models. In particular, it shows that if the asymptotic value of the elasticity of substitution is greater than one, then the marginal product of capital is bounded strictly away from zero. Hence, the aforementioned result regarding the CES is generalized to any production function; namely, an elasticity of substitution between capital and labor that becomes eventually greater than one can counteract the role of diminishing returns to capital and thus can potentially generate unbounded growth even in the presence of non-reproducible factors and the absence of technical progress. This renders factor substitution a powerful engine of growth.

We view this generalization as an important finding, because in a CES world capital and labor are unnecessarily restricted to be equally substitutable at all stages of development. Indeed, Duffy and Papageorgiou (2000) find that the elasticity of substitution is greater than one in a subsample of 21 high-capital countries, whereas it is less than one in a subsample of 23 low-capital countries. This empirical evidence suggests that the substitutability between capital and labor may depend on the stage of economic development

and more precisely on the extent of capital accumulation per worker.

The remainder of the paper is organized as follows. Section 2 reviews briefly the standard convex optimal growth model. Section 3 shows that an elasticity of substitution between capital and labor that becomes asymptotically greater than one is a sufficient condition for a bounded away from zero marginal product of capital and hence for unbounded growth. The paper explores also the converse relationship between the elasticity of substitution and the marginal product of capital. It shows that unbounded growth can occur even if the elasticity of substitution becomes asymptotically equal to one. In this case, however, it will first have to take values that are strictly greater than one. Section 4 concludes the paper.

2. A Convex One-Sector Growth Model

We begin with a brief review of the standard convex one-sector optimal growth model (for details see Jones and Manuelli 1990, 1997). Consider a competitive economy in which agents maximize their lifetime utility

$$U = \int_0^\infty \frac{c_t^{1-\theta} - 1}{1-\theta} \exp(-\eta t) dt, \quad \theta, \eta > 0 \quad (2.1)$$

where c_t denotes consumption at time t and η is the rate of time preference.

Let K and $L \in \mathbb{R}_+$ denote capital and labor, respectively. The technology in this economy is described by the production function $F(K, L)$, which is assumed to exhibit constant returns to scale with respect to K **and** L ; hence, we can define the production function in intensive form $f(k) \equiv F(K/L, 1)$, where $k \equiv K/L$ is the capital-labor ratio. The following assumptions about f are maintained throughout:

- (A1.) $f(0) \geq 0$; furthermore, $\forall k \in \mathbb{R}_+$
- (A2.) f is at least twice continuously differentiable;
- (A3.) $f' > 0$ (strictly increasing);
- (A4.) $f'' < 0$ (strictly concave).

We can write the budget constraint faced by each agent every period as

$$\dot{k} = f(k) - c = rk + w - c, \quad (2.2)$$

where $r (= f'(k))$ and $w (= f(k) - kf'(k))$ denote the competitive interest rate and wage, respectively. It is known (see, for example, Jones and Manuelli 1997) that maximizing

(2.1) subject to (2.2) yields

$$\lim_{t \rightarrow \infty} \frac{\dot{c}}{c} = \lim_{t \rightarrow \infty} \frac{\dot{k}}{k} = \frac{\lim_{k \rightarrow \infty} f'(k) - \eta}{\theta}. \quad (2.3)$$

Thus, if $\lim_{k \rightarrow \infty} f'(k) > 0$ and in particular greater than the rate of time preference, then there will be unbounded growth, despite the presence of non-reproducible factors (namely labor) and the absence of technical progress.¹

3. Endogenous Growth and the Elasticity of Substitution

In this section we provide a link between the limiting value of the elasticity of substitution and a marginal product of capital that is bounded away from zero. In particular, we prove that in an economy with a convex technology and non-reproducible factors, if the elasticity of substitution between capital and labor becomes in the limit greater than one, then the marginal product of capital remains positive as the economy grows (i.e., $\lim_{k \rightarrow \infty} f'(k) > 0$). In view of (2.3), this establishes a causal relationship between the elasticity of factor substitution and endogenous growth.

First, we define the elasticity of substitution $\sigma(k) \equiv -\frac{f'(k)}{kf(k)} \frac{f(k)-kf'(k)}{f''(k)} > 0$ and make the following assumption regarding its limiting behavior:²

$$(A5.) \quad \lim_{k \rightarrow \infty} \sigma(k) = \bar{\sigma} \text{ exists.}$$

Next, we denote the ratio of the the marginal products of labor and capital, which is equal to the absolute value of the slope of an isoquant, as $m(k) \equiv \frac{f(k)-kf'(k)}{f'(k)} > 0$. The function $m(k)$ is increasing and hence we can write $k = h(m)$, where

$$h'(m) = \frac{1}{m'(k)} = -\frac{[f'(k)]^2}{f(k)f''(k)} > 0.$$

¹Naturally, in addition to a bounded interest rate one must impose a condition on the willingness to save. In the Ramsey-Cass-Koopmans model, considered here, this requires imposing an upper bound on the rate of time preference; namely, $\eta < A \equiv \lim_{k \rightarrow \infty} f'(k)$ [see also Condition G in Jones and Manuelli 1990]. In the Solow model, on the other hand, one must impose a lower bound directly on the (exogenous) savings rate, $s: s > \delta/A$, where δ denotes the depreciation rate. Throughout the paper we assume, as it is the case in most of the literature, that these conditions hold and focus instead on the role played by the marginal product of capital and the elasticity of substitution. We also note that in a standard overlapping generations model unbounded growth cannot exist in the presence of a convex technology, such as the one considered here (see Boldrin 1992 and Jones and Manuelli 1992).

²We remind the reader that for a general production function $F(K, L)$ the elasticity of substitution is defined as $\frac{d \ln(K/L)}{d \ln(F_L/F_K)}$ holding output constant. Under constant returns to scale and with two factors of production this can be written as $\frac{F_K F_L}{F F_{KL}}$, which, in terms of the intensive form f , is equal to the one given in the text.

In this notation we can write,

$$\sigma(k) = \frac{h'(m)m}{h(m)} > 0. \quad (3.1)$$

We now establish the first of our two main results.

Theorem 1.

$$\lim_{k \rightarrow \infty} \sigma(k) > 1 \Rightarrow \lim_{k \rightarrow \infty} f'(k) > 0.$$

Proof. Let $k_1 > 0$ be such that $\sigma(k) > \gamma \forall k > k_1$, where $\gamma \in (1, \bar{\sigma})$. Then for every $k > k_1$

$$\int_{m_1}^m \frac{\sigma(x)}{x} dx > \int_{m_1}^m \gamma \frac{1}{x} dx,$$

where $k_1 = h(m_1)$. Or, using (3.1),

$$\int_{m_1}^m \frac{h'(x)}{h(x)} dx > \gamma \ln \left[\frac{m}{m_1} \right],$$

or

$$\frac{k}{k_1} > \left(\frac{m}{m_1} \right)^\gamma,$$

and finally

$$m < \Gamma k^{\frac{1}{\gamma}}, \quad (3.2)$$

where $\Gamma \equiv m_1 k_1^{-\frac{1}{\gamma}}$. Next note that, in a diagram where K is measured on the vertical axis and L on the horizontal, $\Gamma k^{\frac{1}{\gamma}} > 0$ is the absolute value of the slope of an isoquant of a CES production function with an elasticity of factor substitution equal to γ (Lemma 1 in the Appendix). Furthermore, if $\gamma > 1$ then a CES isoquant intersects the K axis. It follows then from (3.2) that the same is true for an isoquant of a production function with an elasticity that is in the limit greater than γ , since the latter is flatter than the former. Without loss of generality consider a unit-isoquant $\{(L, K) \in R_+^2 \mid L = \frac{1}{f(k)}\}$. Then $\lim_{L \rightarrow 0} K = \lim_{L \rightarrow 0} kL = \lim_{k \rightarrow \infty} \frac{k}{f(k)} = \lim_{k \rightarrow \infty} \frac{1}{f'(k)}$, where the last equality follows from L' Hospital's rule. Since there is a vertical intercept, $\lim_{L \rightarrow 0} K$ is finite and positive, which implies that $\lim_{k \rightarrow \infty} f'(k) > 0$.³ ■

Intuitively, unbounded growth occurs because a technology with an elasticity of substitution, $\sigma \equiv \frac{d \ln(K/L)}{d \ln(F_L/F_K)}$, greater than one can accommodate large changes in the capital-labor ratio with small changes in the wage-rental ratio.

³We note that the existence of $\lim_{k \rightarrow \infty} \sigma(k)$ is not required. That is, following basically the same steps one can show that if there exists k_1 such that for every $k > k_1$, $\sigma(k) > 1$ then $\lim_{k \rightarrow \infty} f'(k) > 0$.

Examples. Some well-known production functions for which the elasticity of substitution is in the limit greater than one and hence can lead to unbounded growth, even in the absence of technical progress, are the following:

a. CES. Consider the CES production function

$$f(k) = A [\alpha + \beta k^{-\rho}]^{-\frac{1}{\rho}}, \quad A, \alpha, \beta > 0, \quad \rho > -1,$$

where $\sigma = \frac{1}{1+\rho}$. If $\rho < 0$, then

$$\sigma(k) = \frac{1}{1+\rho} \in (1, \infty) \quad \text{and} \quad \lim_{k \rightarrow \infty} f'(k) = A\beta^{-\frac{1}{\rho}} > 0.$$

It is well known, since Solow's seminal contribution (Solow 1956), that a CES production function with $\sigma > 1$ can lead to unbounded growth even in the absence of technical change. One can also see this from equation (2.3).

b. VES. Consider the following variable elasticity of substitution (VES) production function (see Revankar 1971):

$$f(k) = Ak^\alpha [1 + \beta\alpha k]^{1-\alpha}, \quad A > 0, \quad \alpha \in [0, 1], \quad \beta > -1,$$

where $\sigma = 1 + \beta k$. If $\beta > 0$, then

$$\lim_{k \rightarrow \infty} \sigma(k) = \infty \quad \text{and} \quad \lim_{k \rightarrow \infty} f'(k) = A(\beta\alpha)^{1-\alpha} > 0.$$

It follows that if $\beta > 0$ and the appropriate condition on ρ holds (see also footnote 1) then there will be unbounded growth.

c. Jones and Manuelli. Finally, consider the following production function, which can result in endogenous growth (see Jones and Manuelli 1990 and equation (2.3)):

$$f(k) = Ak + Bk^\alpha, \quad A, B > 0, \quad \alpha \in [0, 1],$$

where $\sigma = \frac{Ak + \alpha Bk^\alpha}{\alpha(Ak + Bk^\alpha)}$. For this production function

$$\lim_{k \rightarrow \infty} \sigma(k) = \frac{1}{\alpha} > 1 \quad \text{and} \quad \lim_{k \rightarrow \infty} f'(k) = A > 0.$$

The same result holds even in the more general case considered in Jones and Manuelli (1990, 1997), where $f(k) = Ak + g(k)$ and $\lim_{k \rightarrow \infty} g'(k) = 0$.

Next we clarify further Theorem 1 with the following two remarks. The first remark shows that for unbounded growth to emerge the elasticity of substitution need not be

everywhere greater than one; it suffices to be greater than one only in the limit. The second remark provides an example of a production function where the elasticity of substitution is greater than one for every finite value of the capital stock and is equal to one only in the limit; nevertheless, such a production function does not lead to unbounded growth.

Remark 1. Let $f(k) = Ak + 1 - e^{-\beta k}$, where $A, \beta > 0$. Then $\sigma(k) = \frac{A + \beta e^{-\beta k}}{Ak + 1 - e^{-\beta k}} \frac{1 - e^{-\beta k}(1 + \beta k)}{\beta^2 k e^{-\beta k}}$, $\lim_{k \rightarrow \infty} \sigma(k) = \infty$, $f'(k) = A + \beta e^{-\beta k}$ and $\lim_{k \rightarrow \infty} f'(k) = A > 0$. Hence, this production function can lead to unbounded growth, despite the fact that for appropriately selected values of A and β , $\sigma(k)$ is less than 1 for low values of k .

Remark 2. Let $f(k) = Ak^\alpha - \gamma$, where $\alpha \in (0, 1)$, known as a the constant marginal shares production function. Then

$$\sigma(k) = 1 - \frac{\alpha\gamma}{1 - \alpha} \frac{1}{f(k)} > 1 \quad \text{if } \gamma < 0 \text{ and } \lim_{k \rightarrow \infty} \sigma(k) = 1.$$

Substituting in the appropriate formulas yields $\lim_{k \rightarrow \infty} f'(k) = 0$, and hence there is no unbounded growth.

Next, we examine the converse relationship between $\lim_{k \rightarrow \infty} \sigma(k)$ and $\lim_{k \rightarrow \infty} f'(k)$. Consider first the following Lemma.

Lemma 2.

$$\lim_{k \rightarrow \infty} f'(k) > 0 \Rightarrow \lim_{k \rightarrow \infty} \frac{f'(k)k}{f(k)} = 1.$$

Proof: See the Appendix. ■

Intuitively, as k increases the marginal ($f'(k)$) and the average product ($f(k)/k$) of capital, which in the limit are equal to each other, approach a positive constant. Given that markets are competitive and thus capital is paid its marginal product, it follows that as k increases without bound all income goes to capital, i.e., $\lim_{k \rightarrow \infty} f(k) = \lim_{k \rightarrow \infty} [f(k)/k]k = \lim_{k \rightarrow \infty} f'(k)k$.⁴

Theorem 2. a)

$$\lim_{k \rightarrow \infty} f'(k) > 0 \Rightarrow \lim_{k \rightarrow \infty} \sigma(k) \geq 1.$$

b) There exists at least one interval (k', k'') such that if $\lim_{k \rightarrow \infty} f'(k) > 0$ then $\sigma(k) > 1$ for every $k \in (k', k'')$.

⁴For a discussion on possible extensions and reinterpretations of the model so that to avoid the undesirable property that labor's share in income converges to zero see Jones and Manuelli (1997, pp. 84-87).

Proof. a) Recall that $m(k) \equiv \frac{f(k)-kf'(k)}{f'(k)} > 0$. Let $\phi(k) \equiv \ln m(k)$. Then

$$\sigma(k) = \frac{1}{\phi'(k)k} > 0. \quad (3.1')$$

Next notice that

$$\lim_{k \rightarrow \infty} \frac{e^{\phi(k)}}{k} = \lim_{k \rightarrow \infty} \frac{m(k)}{k} = \lim_{k \rightarrow \infty} \frac{f(k)}{f'(k)k} - 1 = 0,$$

where the last equality follows from Lemma 2. Since $\lim_{k \rightarrow \infty} \frac{e^{\phi(k)}}{k} = 0$, there exists $k_2 > 0$ such that $\frac{e^{\phi(k)}}{k} < 1 \forall k > k_2$, or $\phi(k) < \ln k \forall k > k_2$. From this and (3.1'), it follows that

$$\int_{k_2}^k \frac{1}{x\sigma(x)} dx < \int_{k_2}^k \frac{1}{x} dx + c',$$

where $c' > 0$ results from the combination of the two constants of integration. Furthermore, since $\bar{\sigma} - \varepsilon < \sigma(k) < \bar{\sigma} + \varepsilon$, $\varepsilon > 0$, the last expression implies

$$\left[\frac{1}{\bar{\sigma} + \varepsilon} - 1 \right] \int_{k_1}^k \frac{1}{x} dx < c',$$

which, in turn yields, implies that $\lim_{k \rightarrow \infty} \sigma(k) \geq 1$.

b) Using the approach followed in the proof of Theorem 1, one can show that

$$\text{there exists } k_3 \text{ such that } \sigma(k) \leq 1 \forall k > k_3 \Rightarrow \lim_{k \rightarrow \infty} f'(k) = 0.$$

The contrapositive of this statement is

$$\lim_{k \rightarrow \infty} f'(k) > 0 \Rightarrow \forall k_3 \text{ there exists } k_4 > k_3 \text{ such that } \sigma(k_4) > 1,$$

and by continuity the result follows. ■

The meaning of Theorem 2 is that a technology with a bounded marginal product of capital will exhibit an elasticity of factor substitution whose limit is greater than or equal to one. Even if the limit is equal to one, however, the elasticity cannot approach one from below, but rather it will take first some values that are greater than one. The following example illustrates this case.

Example. We seek an example of a production function such that the elasticity of substitution is in the limit equal to one, but the marginal product of capital is bounded away from zero. Recall that $\sigma \equiv \frac{dk/k}{dm/m}$, where $m \equiv \frac{f(k)}{f'(k)} - k$. Integrating we have

$$m = \alpha \exp \left(\int \frac{d \ln k}{\sigma(k)} \right),$$

where α is positive constant of integration. Substituting the definition of m and integrating again yields:

$$f(k) = A \exp \left\{ \int \frac{1}{k + \alpha \exp \left(\int \frac{d \ln k}{\sigma(k)} \right)} dk \right\}, \quad (3.3)$$

where A is another positive constant of integration. Equation (3.3) suggests the following functional form: $\sigma(k) = 1 + b/\ln k$, $b > 0$, which approaches one from above.⁵ After integration we have

$$f(k) = A \exp \left(y - \alpha \int \frac{1}{y^b + \alpha} dy \right), \quad \text{where } y = \ln k + b. \quad (3.4)$$

Setting $b = 1$ in (3.4) yields⁶

$$f(k) = A \exp(1 + \ln k) (\alpha + 1 + \ln k)^{-\alpha}$$

and hence $\lim_{k \rightarrow \infty} f'(k) = 0$. Setting $b = 2$, however, results in

$$f(k) = A \exp(2 + \ln k) \exp \left(-\alpha^{1/2} \arctan \frac{2 + \ln k}{\alpha^{1/2}} \right),$$

which after differentiating and taking limits yields

$$\lim_{k \rightarrow \infty} f'(k) = A \exp \left(-\frac{\alpha^{1/2}}{2} \pi + 2 \right) > 0.$$

4. Conclusion

This paper has investigated the role of the elasticity of factor substitution in one-sector convex growth models in which there are present non-reproducible factors, such as labor, and hence there exist diminishing returns to scale with respect to the capital stock (the augmentable factor). It has established a connection between the existence of a lower bound on the marginal product of capital and the asymptotic value of the elasticity of substitution between capital and labor. Accordingly, it has documented that an elasticity of substitution that is only in the limit greater than one (joined with an accommodating

⁵Functions of the form $1 + b/k^n$, $b, n > 0$, either yield $\lim_{k \rightarrow \infty} f'(k) = 0$ or do not yield an explicit functional form for the production function.

⁶The reader can easily verify that if $b = 0$, then (3.4) gives the Cobb-Douglas production function.

rate of time preference or savings rate) is sufficient for unbounded endogenous growth. Furthermore, it has explored the converse relationship and has shown that an elasticity of substitution that takes first some values that are greater than one and becomes in the limit equal to one might yield unbounded growth.

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Appendix

Lemma 1. An isoquant with slope $-\Gamma k^{1/\gamma}$, $\Gamma, \gamma > 0$, is generated by a CES production function with elasticity of substitution equal to γ .

Proof. The equation that describes an isoquant is $Lf(k) = c'$, where c' is a constant. Set the slope of the isoquant, $dK/dL = -[f(k) - f'(k)k]/f'(k)$, equal to $-\Gamma k^{1/\gamma}$; thus,

$$\frac{f(k) - f'(k)k}{f'(k)} = \Gamma k^{1/\gamma},$$

or

$$\frac{f'(k)}{f(k)} = \frac{1}{k + \Gamma k^{1/\gamma}}. \quad (\text{A1})$$

Suppose first that $\gamma = 1$. Integrating (A1) and denoting with A the constant of integration we get

$$f(k) = Ak^{\frac{1}{\Gamma+1}},$$

which is the equation for the Cobb-Douglas production function ($\sigma(k) = \gamma = 1$).

Next consider the case where $\gamma \neq 1$. Integrating (A1), we get

$$f(k) = A \left(\Gamma + k^{\frac{\gamma-1}{\gamma}} \right)^{\frac{\gamma}{\gamma-1}},$$

which is the equation for a CES production function with $\sigma = 1/[1 + (1 - \gamma)/\gamma] = \gamma$. ■

Proof of Lemma 2. The function $f(k)/k$ is non-increasing and positive; hence, it converges. By the Mean Value Theorem there exists $\tilde{k} \in (k, 2k)$ such that

$$\frac{f(2k) - f(k)}{2k - k} = f'(\tilde{k}), \quad \text{or equivalently,} \quad \frac{2f(2k)}{2k} - \frac{f(k)}{k} = f'(\tilde{k}).$$

Taking limits yields

$$\lim_{k \rightarrow \infty} \frac{f(k)}{k} = \lim_{k \rightarrow \infty} f'(k), \quad \text{and since } \lim_{k \rightarrow \infty} f'(k) > 0 \text{ we have } \lim_{k \rightarrow \infty} \frac{f'(k)k}{f(k)} = 1.^7 \quad \blacksquare$$

⁷Alternatively, since $\lim_{k \rightarrow \infty} f'(k) > 0$, it follows that $\lim_{k \rightarrow \infty} f(k) = \infty$. Hence one can apply L'Hospital's rule to get $\lim_{k \rightarrow \infty} \frac{f(k)}{k} = \lim_{k \rightarrow \infty} f'(k)$.