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Seasonal Mackey-Glass-GARCH Process and Short-Term Dynamics

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Abstract

The aim of this article is the study of complex structures which are behind the short-term predictability of stock returns series. In this regard, we employ a seasonal version of the Mackey-Glass-GARCH(p,q) model, initially proposed by Kyrtsou and Terraza (2003) and generalized by Kyrtsou (2005, 2006). It has either negligible or significant autocorrelations in the conditional mean, and a rich structure in the conditional variance. To reveal short or long memory components and non-linear structures in the French Stock Exchange (CAC40) returns series, we apply the test of Geweke and Porter-Hudak (1983), the Brock et al. (1996) and Dechert (1995) tests, the correlation-dimension method of Grassberger and Procaccia (1983), the Lyapunov exponents method of Gençay and Dechert (1992), and the Recurrence Quantification Analysis introduced by Webber and Zbilut (1994). As a confirmation procedure of the dynamics generating future movements in CAC40, we forecast the return series using a seasonal Mackey-Glass-GARCH(1,1) model. The interest of the forecasting exercise is found in the inclusion of high-dimensional non-linearities in the mean equation of returns.

Key Words: Noisy chaos, short-term dynamics, correlation dimension, Lyapunov exponents, recurrence quantifications, forecasting.

JEL classification: C49, C51, C52, C53, D84, G12, G14.

1. Introduction

According to the efficient financial markets hypothesis (henceforth EMH) price changes are a result of the availability of new information and such information is rapidly impounded into the new price. EMH has been divided into three forms, defined in terms of specified categories of information. The weak form tests the random walk model itself, using autocorrelation tests and run analysis to investigate whether past prices indicate anything about futures prices. Semi-strong form testing investigates whether publicly available information other than prices is reflected in prevailing prices, and strong-form testing investigates whether private information is reflected in prevailing prices.

Financial markets are highly complex feedback systems in which all the investors together overreact to information or withhold action in the face of information. These feedback processes are non-linear because they are based on a non-proportional cause-effect relationship. The presence of non-linear dependence in financial markets undercuts the Fama's (1965) EMH, which is founded on assertions about the independence of successive stock price changes. Non-linear dependence suggests that there are deeper structural forces, possibly including noisy chaotic dynamics¹ that affect financial markets outcomes. The advantage of a noisy chaotic perspective is that it considers structural factors jointly with external noise for explaining markets fluctuations, although for traditional stochastic theory only exogenous forces exist.

The EMH in finance is closely related to the rational expectations hypothesis in economics. In an efficient market framework agents try to maximise their expected returns; given any observed information, all agents agree on the mean interpretation of such information (homogenous agents). Agents are supposed to be rational and this is common knowledge. All financial risks and observed volatility arise from causes which are external to the information system. In a rational expectation model, market equilibrium equations are usually assumed to be in the information set, and agents use these underlying market equilibrium equations in forming their rational expectations forecast.

The conclusions of the theory of rational expectations are contradicted by many empirical observations and common experience of markets agents. Indeed, the implications of the theory have been rejected in broad areas of economics. According to the recent findings of Hristu-Varsakelis and Kyrtsov(2008), Kyrtsov and Vorlow (2005, 2009), Kyrtsov and Labys (2006, 2007), Kyrtsov and Serletis (2006) and Kyrtsov and Malliaris (2006) among others, fluctuations in macroeconomic and financial series are

driven by noisy non-linear deterministic dynamics. In reality, traders face different transaction costs, have different information sets, work with different time scales and time horizons, and have different expectations about future dividends and stock prices. When heterogeneity and non-linear trading strategies are taken into account, we can obtain results certainly different from those of the homogenous agent rational expectations models, and thus we can be closer to the empirical properties of stock series, i.e. fat tails, volatility clustering etc.

Brock and Hommes (1998), Hommes (2006), Lux (1995, 1998), Chen et al. (2001), Malliaris and Stein (1999), Gaunersdorfer (2000, 2001), and Chiarella et al. (2002) show that due to heterogeneity in expectations, agent-based non-linear financial models produce chaotic dynamics. Non-linear dynamic models can generate a wide variety of irregular patterns. A non-linear chaotic model, buffeted with dynamic noise, with some autocorrelations in returns but at the same time persistence in squared returns, with slowly decaying autocorrelations, may provide a structural explanation of the unpredictability of stock returns and volatility clustering².

The main objective of this article is the study of complex structures which are behind the short-term predictability of stock returns series. However, in the present work we are not going to compare the forecasting performance of various models. We are rather interested in emerging the causes of the resulting dynamics. In this regard, we filter the complex underlying structures using a seasonal version of the noisy chaotic Mackey-Glass-GARCH(p,q) model initially proposed by Kyrtsou and Terraza (2003), that can present either negligible or significant autocorrelations in the conditional mean, and a rich structure in the conditional variance. Using noisy chaotic modelling for forecasting financial time series is an attractive attempt, because of its unique feature to combine stochastic dynamics and pure deterministic behaviour.

To reveal short or long memory components and non-linear structures in the French Stock Exchange (CAC40) returns series, we apply the test of Geweke and Porter-Hudak (1983), the Brock et al. (1996) (henceforth BDS) and Dechert (1995) tests, the correlation-dimension method of Grassberger and Procaccia (1983), the Lyapunov exponents method of Gençay and Dechert (1992), and the Recurrence Quantification Analysis introduced by Webber and Zbilut (1994). As a confirmation procedure of the dynamics generating future movements in CAC40, we perform out-of-sample forecasts of the CAC40 returns series using a seasonal Mackey-Glass-GARCH(1,1) model. The interest of the forecasting

¹ We are interested in the case of deterministic chaotic dynamics that are perturbed by additive normal or heteroskedastic noise.

² According to the volatility-clustering phenomenon stock-price fluctuations are characterised by episodes of low volatility with small price changes irregularly mixed by episodes of high volatility with large price changes.

exercise is found in the inclusion of high-dimensional non-linearities in the mean equation of returns. For reasons of presentation simplicity, we prefer to report directly the empirical results in section 3, after a brief description of the Recurrence Quantification Analysis in section 2. A detailed presentation of the rest of techniques can be found in Kyrtsou and Terraza (2002, 2003) and Kyrtsou et al. (2004). Section 4 concludes the paper.

2. Recurrence Quantification Analysis

Recurrence quantification analysis (RQA) is an extension of recurrence plots (RPs), first introduced by Eckmann et al. (1987). With RP, one can graphically detect hidden patterns and structural changes in data or see similarities in patterns across the time series under study. The fundamental assumption underlying the idea of the RPs is that an observable time series is the realisation of some dynamical process, the interaction of the relevant variables over time. For example in a stock market, prices are determined by many factors, such as the economic and political environment, investors' expectations and individual traders' decisions etc. In this case, we have a multidimensional dynamical system and one-dimensional output, which is the series of scalar observations.

RQA is a relatively new analytical tool for the study of non-linear dynamical systems developed by Webber and Zbilut (1994) and then applied to theoretical time series by Trulla et al. (1996) and Zbilut et al. (2000). RQA methodology can be summarised as follows: the embedding matrix corresponding to the studied series is constructed by the method of time delays (Takens, 1981). Thus, $x_t^m = (x_t, x_{t+\tau}, \dots, x_{t+(m-1)\rho})$ are the artificial vectors, where $t = 1, \dots, T-(m-1)\rho$, T is the number of observations, m is the embedding dimension, and ρ is the time delay. Next, using a Euclidean norm, distances D in n -space between individual i - j pairs are calculated. A RP is a graphical representation of the distances matrix $D_{i,j}$, by darkening the point at coordinates (i,j) that corresponds to a distance value between i and j vectors lower than a predetermined critical radius ϵ . The plot is symmetric ($D_{i,j}=D_{j,i}$) and the main diagonal is always darkened ($D_{i,j}=0, i=j$). The main feature of RPs is that if the series is fully deterministic, the system's attractor will be revisited by the trajectory sometime in the future. In that case, the RP will show short line segments parallel to the main diagonal, as for the Lorenz system (Figure 1). If the time series is independent and identically distributed, as a white noise, then the RP will not display any kind of structure (Figure 2).

Unfortunately, recurrence plots give results that are very difficult to be evaluated quantitatively. For this reason, RQA was developed to provide quantification of important aspects revealed through the plot. Thus, the number of contiguous diagonal points, which constitute line segments are tallied as a percent

of recurrent points, i.e. %determinism. Furthermore, we calculate the Shannon entropy of line segment distributions. In general, entropy values are low within periodic windows (epochs), because all lines are of identical length. Contrariwise, entropy values are high within chaotic (non-periodic) windows because there is a large diversity in diagonal line lengths. Some applications of the RQA in macroeconomic and financial time series can be found in Kyrtsov and Vorlow (2005), Strozzi et al. (2002), Belaire-Franch (2004), and Belaire-Franch et al. (2002).

Figure 1: Recurrence plot for the Lorenz system

Figure 2: Recurrence plot for a White Noise

3. Empirical results

3.1 Data and preliminary tests results

The data used here are the daily index series (CAC40) of the French Stock Exchange, during the period 09/11/1987-05/28/1999, giving 3056 observations. The augmented Dickey-Fuller unit-root test shows that one unit root exists in the CAC40 series. So, the data are log-differenced to give DLCAC40. The BDS and Dechert (1995) procedures test for the presence of dependence. Their tests statistics are defined as follows:

$$W^{BDS} = \sqrt{T} \frac{D_{m,T}(\epsilon) - (D_{1,T}(\epsilon))^m}{\sigma_{m,T}(\epsilon)} \xrightarrow{d} N(0,1)$$

$$W^D = \sqrt{T} \frac{S_{m,T}(\epsilon_1, \epsilon_2)}{\sigma_T(\epsilon_1, \epsilon_2)} \xrightarrow{d} N(0,1)$$

which means that W^{BDS} and W^D converge in distribution to $N(0,1)$, where $S_{m,T}(\varepsilon_1, \varepsilon_2) = C_{m,T}(\varepsilon_1, \varepsilon_2) - C_T(\varepsilon_1)C_T(\varepsilon_2)$. Here $\sigma_{m,T}^2(\varepsilon)$, $\sigma_T^2(\varepsilon_1, \varepsilon_2)$, and $D_{m,T}(\cdot)$, $C_{m,T}(\cdot)$ are reciprocally the asymptotic variances and the correlation integrals, given by Brock et al. (1992), Brock et al. (1996) and Dechert (1995). Also m is the embedding dimension, ε is the radius and σ is the standard deviation of the returns. In our application we vary m from 2 to 5, and we use $\varepsilon=0.5\sigma$, 1σ , 1.5σ , 2σ for the BDS test and $\varepsilon_1=\varepsilon$, and $\varepsilon_2=2\varepsilon_1$ for the Dechert test.

The results for the CAC40 returns, reported in Tables 1a and 1b, reveal that the null of i.i.d. is rejected with a $|W^{BDS}| > 1.96$ and $|W^D| > 1.96$ for $\alpha=5\%$.

Table 1a: BDS test results for the DLCAC40 series

ε/σ	0.5	1	1.5	2
$m = 2$	4.2522	5.6170	7.6004	9.0979
$m = 3$	5.5562	7.3283	9.8005	12.056
$m = 4$	6.9679	8.9349	11.376	13.817
$m = 5$	8.7084	10.235	12.588	14.822

Table 1b: Dechert test results for the DLCAC40 series

ε_1 and $\varepsilon_2=2\varepsilon_1$	0.5σ	1σ	1.5σ	2σ
$m = 2$	4.8359	7.882	10.76	11.935
$m = 3$	6.1605	8.177	9.4596	9.6429
$m = 4$	5.5581	6.6106	8.0336	8.6279
$m = 5$	5.0039	6.7708	8.7819	9.4825

Leptokurtosis in the data is revealed by high kurtosis coefficient of 9.2012 and a Jarque-Bera statistics of 4,996.97. The Engle (1982) test result confirms the presence of heteroskedasticity ($TR^2=71.27 > \chi^2(1)$). Table 2 presents the spectral regression estimates of the Geweke and Porter-Hudak (1983) fractional-differencing parameter d over the in-sample period. If the calculated t-statistics is superior to 1.96 for $\alpha=5\%$, then we accept the hypothesis that $d \neq 0$, i.e. we have either an anti-persistent or persistent process. Thus, as Table 2 reports, there is strong evidence that the CAC40 returns exhibit short memory, since the t-statistics value is widely inferior to 1.96. Nevertheless, we cannot definitely conclude that an ARCH process generates the CAC40 series. It has been shown that the fractional-integration coefficient can be identical for both chaotic and ARCH series (for more details see Kyrtsou and Terraza (2002, 2003)). Geweke (1993) underlined that simple maximum likelihood can face problems when analysing series with underlying chaotic structures.

Table 2: Fractional-integration coefficient d for the DLCAC40 series

Method	d value	Standard error	t-statistics	p-value
<i>GPH</i>	0.209491	0.316536	0.610911	0.530363

Also interesting is the behaviour of the correlation-dimension estimator for noisy chaotic and pure stochastic series. In both cases the correlation dimension is very high. This makes more difficult the identification procedure. More specifically for the DLCAC40 series the correlation dimension is equal to 6.522. Taking into account the constraints of applying such methods to short and noisy time series, we should reinforce the obtained results with additional empirical testing. To get a more complete description of the dynamic behaviour, we compute the Lyapunov exponents of the DLCAC40, which is a powerful tool for distinguishing between low-dimensional chaos and stochastic processes. However, in the presence of dynamical noise the algorithm developed by Wolf *et al.* (1985), which is used to estimate the growth rates of the propagation of small perturbations in the initial conditions, risks to lead to inconsistent conclusions.

To avoid biased estimators due to noise, we apply the algorithm proposed by Gençay and Dechert (1992) based on feedforward neural networks. The Lyapunov exponent for the first input $\lambda(1)$ ($0.2954e-03$) is positive and for the second $\lambda(2)$ (-0.9702) negative. Therefore, we conclude there is not clear evidence for a pure stochastic process. Similar results are obtained in Gençay and Liu (1997) for a noisy logistic map, a noisy Hénon map, and a noisy Mackey-Glass delay equation. With the addition of Gaussian noise, the largest Lyapunov exponent becomes strongly negative. The slight positivity of $\lambda(1)$ could be fed by an underlying high-dimensional chaotic system, which generates similar random behaviour because of hidden dimensionality. The specific shortcomings of Lyapunov exponent algorithms due to noise in data are discussed in details by Dechert and Gençay (1996). Useful discussion can also be found in Bask and Gençay (1998) and Dechert and Cencay (2000).

Although several methods for dependence and chaotic structures are employed, we are not able to obtain clear conclusions regarding the nature of the dynamics. A possible explanation is that financial series may include both chaotic and heteroskedastic structures. To test this hypothesis, we enrich the mean equation of our model with Mackey-Glass dynamics, accompanying by a traditional GARCH process for the variance equation.

3.2 The Mackey-Glass-GARCH(1,1) model

As for several stock indices, the autocorrelations of the Paris Stock Exchange returns series are not significant, suggesting the presence of white noise structure. Therefore, the results of the BDS test, presented in the previous section, show that the CAC40 returns series is not an i.i.d. process.

The deterministic part of the Mackey-Glass-GARCH model is a discretized variant of the chaotic Mackey-Glass delay equation (Mackey and Glass, (1977)) plus noise:

$$X_t = a \frac{X_{t-\tau}}{1 + X_{t-\tau}^c} - (b-1)X_{t-1} + \varepsilon_t = a \frac{X_{t-\tau}}{1 + X_{t-\tau}^c} - \delta X_{t-1} + \varepsilon_t \quad (1)$$

α , b and δ are parameters to be estimated. τ is the delay and c a constant. The advantage of this model is that it can filter various types of non-linear dependencies in the mean with the simple increase of c and τ . This discrete version of the deterministic Mackey-Glass equation has previously been used in neural network literature (Gallant and White (1992), Goffe et al., (1994), Gençay and Liu (1997)).

Applying the Engle (1982) test to the residuals of the noisy Mackey-Glass equation confirms the existence of heteroskedasticity ($TR^2=39.31 > \chi^2(1)$). Hence, the conditional variance can be given as follows:

$$h_t = \alpha_0 + \alpha_1 \varepsilon_{t-1}^2 + \beta_1 h_{t-1}$$

and $\varepsilon_t | I_t \sim N(0, h_t)$

The optimal c and τ are chosen on the basis of Log Likelihood and Schwarz criteria. In this case $c=2$ and $\tau=1$ are selected. This model is a simpler version of the generalized Mackey-Glass-GARCH process developed by Kyrtsou (2005, 2006) and applied in Kyrtsou and Serletis (2006). The main characteristic of the non-linear trading strategy in the mean equation of the above model is that it can take into account dynamics produced by both positive and negative feedback traders. The coefficients a and δ vary over time. When the sum of coefficients a and δ is positive we observe positive feedback behaviour, while negative sum reveals negative feedback (Kyrtsou (2006), Kyrtsou and Labys (2007)). The Mackey-Glass-GARCH model is interesting because of its similarity to economic and financial series; it is non-linear in mean and variance. Its advantage over simple GARCH and AR-GARCH alternatives has been shown in Kyrtsou and Terraza (2003) and Kyrtsou and Karanasos (2006).

Estimation results of the Mackey-Glass-GARCH(1,1) process (henceforth MG-GARCH(1,1)) are given in Table 3. We estimate the model employing the Quasi-Maximum Likelihood estimation. For comparison purposes, we also report estimations for GARCH (1,1) and AR(4)-GARCH(1,1) models.

The best model is chosen on the basis of Likelihood Ratio (LR) and Schwarz Information Criterion (SIC)³.

Table 3: Estimates for the 3 models

Coefficients	MG-GARCH(1,1)	AR(4)-GARCH(1,1)	GARCH(1,1)
PANEL A			
φ_1	-	0.058 (2.96*) ¹	-
φ_2	-	0.008 (0.409)	-
φ_3	-	-0.025 (-1.256)	-
φ_4	-	0.034 (1.95*)	-
a	187,53 (2.55*)	-	-
δ	187,44 (2.54*)	-	-
α_0	0.0000793 (3.16*)	0.00000116 (2.58*)	0.00000114 (5.182*)
α_1	0.14999 (4.13)	0.0929 (4.18*)	0.0921 (4.388*)
β_1	0.5999 (5.36)	0.866 (29.58*)	0.868 (38.725*)
PANEL B			
ML	11874,35	11850.24	11852.63
SIC	-7,7631	-7.656	-7.646
K	6,36	6.50	6.53
JB	1545,8	1618.9	1624.5
Q-stat(1)	0,001 (0,938)	-0.003 (0.865)	0.048 (0.008*)
Q-stat(12)	12.457 (0.410)	0.030 (0.835)	0.028 (0.104)
Q-stat(24)	24.369 (0.441)	0.003 (0.785)	0.006 (0.199)

* : $p < 0.05$ (S)⁴.

¹: in the first panel, within parentheses t-statistics are reported.

²: in the second panel, within parentheses probabilities are reported.

Table 3 reports models estimates as well as their summary statistics. The estimation shows significant improvement in the maximum likelihood (ML) of the MG-GARCH specification over the other two traditional processes. Besides, according to the Schwarz information criterion (SIC), in all cases the optimal model is again the nonlinear MG specification. The choice of the MG model is also supported by the application of two more tests for nonlinearity in the mean; the Tsay (1986) and White (1989) tests. Both tests give values that confirm the presence of strong non-linear structure in the mean

³ As it is shown in Conrad and Karanasos (2005a,b) among others, the LR test can be used for model selection. Also, the SIC can be employed to rank various GARCH-type models.

⁴ S., stands for significant at 5%.

equation of the CAC40 returns series⁵. The most amazing thing mentioned at the beginning of this section is that this structure is not revealed when employing common techniques as the autocorrelation of returns. Finally we compare the kurtosis (K) and the value of Jarque-Bera (JB) test statistic of the initial returns series and the standardised residuals for the various models. In more details, for the MG-GARCH process the kurtosis has been reduced from 9.2012 to 6.36 and the Jarque-Bera from 4996.97 to 1545.8, and they are significantly improved in comparison with the results of the other models. To test the robustness of these findings, we apply the BDS⁶ and Dechert tests to the standardised residuals. The results in Table 4a show that dependences have been filtered, while in Table 4b almost all values are superior to 1.96. The Dechert test seems to be more robust in this case.

Table 4a: BDS test results for the residuals of the MG-GARCH(1,1) model

ε/σ	0.5	1	1.5	2
$m = 2$	-1.6481	-1.7333	-1.2340	-0.69934
$m = 3$	-1.4274	-1.3577	-0.5489	0.34005
$m = 4$	-0.86936	-0.6654	0.28898	1.2671
$m = 5$	-0.09587	0.1404	1.0166	1.8823

Table 4b: Dechert test results for the residuals of the MG-GARCH(1,1) model

ε_1 and $\varepsilon_2=2\varepsilon_1$	0.5σ	1σ	1.5σ	2σ
$m = 2$	-0.01976	1.2272	3.0949	4.0686
$m = 3$	1.5017	2.5488	3.6046	5.1728
$m = 4$	2.0115	2.1781	1.8718	2.148
$m = 5$	1.6532	2.2393	2.9662	4.1855

Including additional explanatory variables could lead to a better fitting. One way to improve estimation is filtering seasonalities from variance equation. For this reason, exogenous dummies are considered in h_t . So The MG-GARCH(1,1) model takes the following seasonal form:

$$X_t = a \frac{X_{t-1}}{1 + X_{t-1}^2} - \delta X_{t-1} + \varepsilon_t \quad (2)$$

$$\varepsilon_t | I_t \sim N(0, h_t),$$

$$h_t = \alpha_0 + \alpha_1 \varepsilon_{t-1}^2 + \beta_1 h_{t-1} + \eta_1 d_1 + \eta_2 d_2 + \eta_3 d_3 + \eta_4 d_4$$

⁵ For presentation purposes these results are not given here, but there are available upon request.

⁶ Since we apply the BDS test to standardised residuals, we don't compare $|W^{BDS}|$ to 1.96, but we use the Tables in Brock et al. (1992), page 279.

where d_1 is the dummy for Monday and so on. Only four dummies are used to avoid co-linearity between d_5 and the constant α_0 in the variance equation⁷.

In the estimation of the seasonal MG-GARCH(1,1) model (henceforth SMG-GARCH(1,1)), perceptible changes take place in the case of the first three dummies. The significativity of η_1 means that (most of the time) the opening of the French Stock Exchange is associated with a high volatility. Certain authors explain this phenomenon by the accumulation of news during the week-end. The results are given in Table 5.

Table 5: Estimates for SMG-GARCH(1,1) model for the DLCAC40 series

Coefficient	Value	t-Statistics
a	187.5322	1.95*
δ	187.4471	1.95*
α_0	7.14 e-06	3.73**
α_1	0.149997	5.12**
β_1	0.599983	6.75**
η_1	1.57 e-05	4.83**
η_2	-7.35 e-06	-2.70**
η_3	-5.27 e-06	-3.42**
η_4	-8.03 e-07	-0.48

* : $p > 0.05$ (N.S)

** : $p < 0.05$

As we should expect, the new filtering leads to a significant decrease of kurtosis from 5.652 to 4.540449, and Jarque-Bera from 950.05 to 317.5699. The addition of dummies improves the conditional normality of the residuals (Beller and Nofsinger (1998)).

In an attempt to reduce more kurtosis and Jarque-Bera, we extend the specification in the mean equation by considering a new variable ψ_t that includes deviations from the benchmark white noise model. More specifically, estimation contains two stages. In the first regression of model 2, residuals are recuperated. Then via wavelet denoising⁸, noise is extracted from the residual series and a second new series namely ψ_t is also recuperated (see for example Donoho and Johnstone, 1994)⁹. Afterward, the second stage of estimation starts and the model 3 is regressed, including the variable ψ_t .

⁷ For more details see Bourbonnais and Terraza (1998).

⁸ The general wavelet denoising procedure is as follows: We first apply wavelet transform to the noisy signal to produce the noisy wavelet coefficients. Then, we select appropriate threshold limit at each level and threshold method (hard or soft thresholding) to best remove the noises. In our application soft thresholding is used. Finally, we inverse wavelet transform of the thresholded wavelet coefficients to obtain a denoised signal.

⁹ Alternatively, in case of returns with less important dependencies, the denoising technique can be directly applied to the initial returns series.

$$X_t = a \frac{X_{t-1}}{1 + X_{t-1}^2} - \delta X_{t-1} + \gamma \psi_t + \varepsilon_t \quad (3)$$

$$\varepsilon_t | I_t \sim N(0, h_t),$$

$$h_t = \alpha_0 + \alpha_1 \varepsilon_{t-1}^2 + \beta_1 h_{t-1} + \eta_1 d_1 + \eta_2 d_2 + \eta_3 d_3 + \eta_4 d_4$$

The estimation results of the second seasonal model (equation 3) (henceforth SSMG-GARCH(1,1)) previously given, are presented in Table 6. Looking at Table 6, we observe that the coefficients a , δ , and γ are highly significant. In the variance equation, α_0 , α_1 , β_1 , η_1 , η_2 and η_3 are also significant. With the introduction of the ψ_t component in the mean equation, the coefficient η_1 becomes smaller, while for η_2 and η_3 an increase is observed. The model SSMG-GARCH(1,1) fits well the Paris Stock Exchange returns series. Nonetheless, standardised residuals are still non-Gaussian. Kurtosis is reduced from 4.540449 to 3.7522 and Jarque-Bera from 317.5699 to 78.3427. This is a common result first observed by Bollerslev (1987).

Table 6: Estimates for SSMG-GARCH(1,1) model for the DLCAC40 series

Coefficient	Value	t-Statistics
a	225.6784	3.85**
δ	225.5779	3.85**
γ	0.002219	44.62**
α_0	2.67 e-06	5.24**
α_1	0.147579	5.49**
β_1	0.669355	14.14**
η_1	6.96 e-06	6.51**
η_2	-4.80 e06	-9.69**
η_3	-1.93 e-06	-3.02**
η_4	7.35 e-07	0.90

** : $p < 0.05$

A cross-checking of the resulting series employing the BDS and Dechert tests shows the existence of a remaining structure difficult to be identified. Comparing the values of the W^{BDS} and W^D statistics reported in Tables 4a,b and 7a,b, will demonstrate the sensitivity of the BDS test to seasonality. After filtering the variance by seasonal patterns, BDS can correctly capture dependences. In contrast, the Dechert test remains unaffected by the complexity of the hidden structures in the data.

Table 7a: BDS test results for the standardised residuals of the SSMG-GARCH(1,1) model

ε/σ	0.5	1	1.5	2
$m = 2$	-3.0299	-3.2201	-2.8306	-2.2185
$m = 3$	-3.0957	-3.499	-2.9859	-1.9905
$m = 4$	-2.5944	-3.2803	-2.7705	-1.7607
$m = 5$	-2.7805	-3.428	-2.985	-1.8895

Table 7b: Dechert test results for the standardised residuals of the SSMG-GARCH(1,1) model

ε_1 and $\varepsilon_2=2\varepsilon_1$	0.5σ	1σ	1.5σ	2σ
$m = 2$	-1.3977	-0.7581	0.7548	4.0936
$m = 3$	0.3826	0.4768	1.4001	3.7428
$m = 4$	-1.3626	-1.0665	0.4277	3.1229
$m = 5$	1.1348	1.3523	2.0568	4.4191

We can confirm the results of Tables 7a,b, and identify the remaining structure by applying additionally the recurrence quantification analysis and more precisely the %determinism, and entropy tools. The non-linear variables %determinism, and entropy are plotted in Figures 3, 4, 5 and 6 versus the epoch (window) number. We must choose carefully the RQA parameters in order to assure the robustness of results. The delay parameter ρ needs to be selected so as to avoid false linearisations of the system. In our analysis we set ρ equal to the minimum time delayed mutual information. Nevertheless, as Kantz and Schreiber (1997) underlined, a good estimate of the delay is even more difficult to obtain than a good estimate of the embedding parameter, because we often analyse short and noisy data. The embedding dimension is calculated using the method of false nearest neighbours. Regarding the choice of ε , we use a threshold level of the lower 10% of the mean rescaled distance between all embedded vectors, using a Euclidean norm.

In Figure 3, the graph of the %determinism offers important information about dynamic behaviour of the CAC40 index. Between the 23rd and 115th epochs (1st part) a certain structure appears. On the contrary, between the 115th and 291st epochs (2nd part) it is difficult to distinguish similar patterns. As we can observe in Figure 4, the entropy corresponding to the 1st part of Figure 3 is high. We remind that in general high (low) entropy values are obtained within chaotic (periodic) windows. This is in line with Kyrtsov and Terraza (2002, 2003) conclusion, that CAC40 is governed by underlying complex deterministic dynamics.

Figure 3¹⁰: %determinism for the CAC40 returns. Note that the %determinism is plotted versus the epoch (window) number.

Figure 4: Entropy of line distribution for the CAC40 returns. Entropy is also plotted versus the epoch (window) number.

Figure 5 presents the evolution of %determinism for the standardised residuals of the SSMG-GACRH(1,1) model. Taking into account the results of BDS and Dechert tests (Table 7), we should expect the persistence of dependencies. Nevertheless, the detected dynamics are quite different. More specifically, during certain periods there is not any dependence, while for others deterministic points are found. Also interesting is the entropy of line distributions, as given in Figure 6. Only for certain windows high values are detected. These results lead to the conclusion that standardised residuals seem to follow a stochastic process, since similar behaviour can be observed for ARCH type processes. As an attempt to reveal more qualitative features of the structure in the CAC40 series generating such distortions in the standardised residuals of the SSMG-GACRH(1,1) model we calculate the outliers of the initial returns using the Freedman et al. (1978) definition. Comparing Figures 6 and 7 shows that within intervals where entropy in standardised residuals is high, a few outliers are detected in the CAC40 returns. Could the inclusion of outliers in our approach improve the descriptive statistics of residuals? Further investigation is needed in a future work for this very interesting issue.

¹⁰ RQA parameters for figures 3 and 4: embedding dimension = 6, delay = 2, line definition = 2 points, length of each epoch = 128 points, data shift = 10, radius $\epsilon = 10$.

Figure 5¹¹: %determinism for the SSMG-GARCH(1,1) standardised residuals. %determinism is plotted versus the epoch (window) number.

Figure 6: Entropy of line distribution for the SSMG-GARCH(1,1) standardised residuals. Entropy is plotted versus the epoch (window) number.

Figure 7: Observations of the CAC40 returns for which outliers are detected

In focus to the generating mechanism of future movements in CAC40, a forecasting exercise will follow. Out-of-Sample forecasts will be performed employing the SSMG-GARCH(1,1) model. The estimated parameters $\hat{\alpha}$, $\hat{\delta}$, and $\hat{\gamma}$ are used to calculate the prediction¹²:

¹¹ RQA parameters for figures 4 and 5: embedding dimension = 7, delay = 1, line definition = 2 points, length of each epoch = 128 points, data shift = 10, radius $\epsilon = 10$.

$$X_{t+1} = \hat{a} \frac{X_t}{1 + X_t^2} - \hat{\delta} X_t + \hat{\gamma} \psi_{t+1}$$

As it is shown in Table 8, for each forecast horizon a model is computed and the obtained returns are placed in the respective column. In order to evaluate the quality of models in terms of forecasting accuracy we choose the normalised mean square error (NMSE) index, as used by Farmer and Sidorowich (1987):

$$NMSE = \frac{\sum_{t \in P} (x_t - \hat{x}_t)^2}{\sum_{t \in P} (x_t - \bar{x}_P)^2}$$

where $\hat{x}_t$ is the predicted value of X_t , P represents the samples for out-of-sample prediction, and $\bar{x}_P$ is the mean value of P . We prefer to use NMSE because the division by the estimated variance eliminates the dependence on the range of the data. If $NMSE=0$, the predictions are perfect; $NMSE \geq 1$ indicates that the performance is no better than the mean value predictor. The NMSE values calculated for all models are summarised in Table 8. Since a noisy chaotic system enclosing inherent complexity with short-term predictability is found to be the best fitting of the CAC40 series we should expect to identify similar dynamics in the CAC40 returns as well. As it can be seen, the SSMG-GARCH(1,1) gives the best forecasts only for one or alternatively for four steps ahead¹³.

Based on non-linear methods Kyrtsou and Terraza (2002) have shown the predominant character of the underlying complexity in the determination of short-term predictability of the French stock exchange. This is the key feature of chaotic systems. Even though chaos puts a fundamental limit to long-term prediction, it suggests the possibility of short-term prediction based on the fact that random-looking data may contain non-linear deterministic underlying relationships. As Kaplan and Glass (1995) pointed out, when data are generated by complex systems the presence of either dynamic or measurement noise can cause divergence of the predictions from the true values. They report that "...even when the predictions are good, they are not perfect because of the measurement noise". See also Hommes and

¹² Our implementation follows the forecasting scheme for non-linear deterministic processes described in Kaplan and Glass (1995) and Baumol and Quandt (1995).

¹³ These results remain unchangeable even if the number of out-of-sample entries is expanded from 50 to 200. Usually, in that kind of forecasting exercises the number of out-of-sample observations is in the range of 10 to 20 percent of the overall data set. Nevertheless, as Gençay and Liu (1996) underline, the interaction between the non-linear determinism of the underlying mechanism generating the returns with high levels of dynamical noise can deteriorate the performance of the forecasting method. When similar dynamics prevail, only short-term forecasts are possible.

Manzan (2006) and Kyrtsov and Malliaris (2006) for discussion about the impact of noise and interactions with non-linear skeletons.

4. Conclusions

In the present work we tried to investigate whether the generating mechanism of the CAC40 returns is driven by noisy chaotic dynamics, with the aim to determine the nature of stock returns predictability. Following the methodology described in previous sections it is found that the DLCAC40 does not follow an i.i.d process, exhibits short-term memory and it is rather generated by a noisy chaotic process. On the basis of the above, the SSMG-GARCH(1,1) model is fitted to the DLCAC40. Residual testing has revealed that the suggested noisy chaotic GARCH model performs better than pure stochastic alternatives. The use of seasonal dummies in the variance equation can improve the performance of the noisy Mackey-Glass model. The forecasting exercise has shown that the SSMG-GARCH(1,1) model gives best forecasts for one or eventually four steps ahead. Short-term dynamics are indeed the prevailing type of dynamics in the Paris stock market.

The inclusion of non-linear trading strategies in the mean equation of a mixed non-linear process can answer to the crucial question “what is the type of investors, able to create inherent complexity in financial markets which in turn will determine the short-term predictability of stock returns?”. It seems quite evident that the capture of hidden endogenous patterns in stock markets can lead to significant improvements when attempting to forecast stock returns.

The information that a stock return can be predicted over a few steps in the future can reveal characteristics both for the financial market where the stock is traded and the overall macroeconomic environment where the specific market is embedded. Quantifying thus the source and nature of the observed dynamics does not only constitute a pure mathematical way of approaching phenomena. It is still more a pertinent method for recovering and understanding economic behaviour, as well as relationships between economic variables and systems.

A possible extension to this study is to consider more than one group of feedback traders in the mean equation of the SSMG-GARCH(1,1) model and in a multivariate framework compare non-linear dynamics between different stock markets.

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